Free-Electron Laser Theory for Coherent Electron Cooling

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Outline

- Intra-beam Scattering
- Coherent Electron Cooling & Debye Screening
- Three-Dimensional FEL Theory
- FEL Dispersion Relation

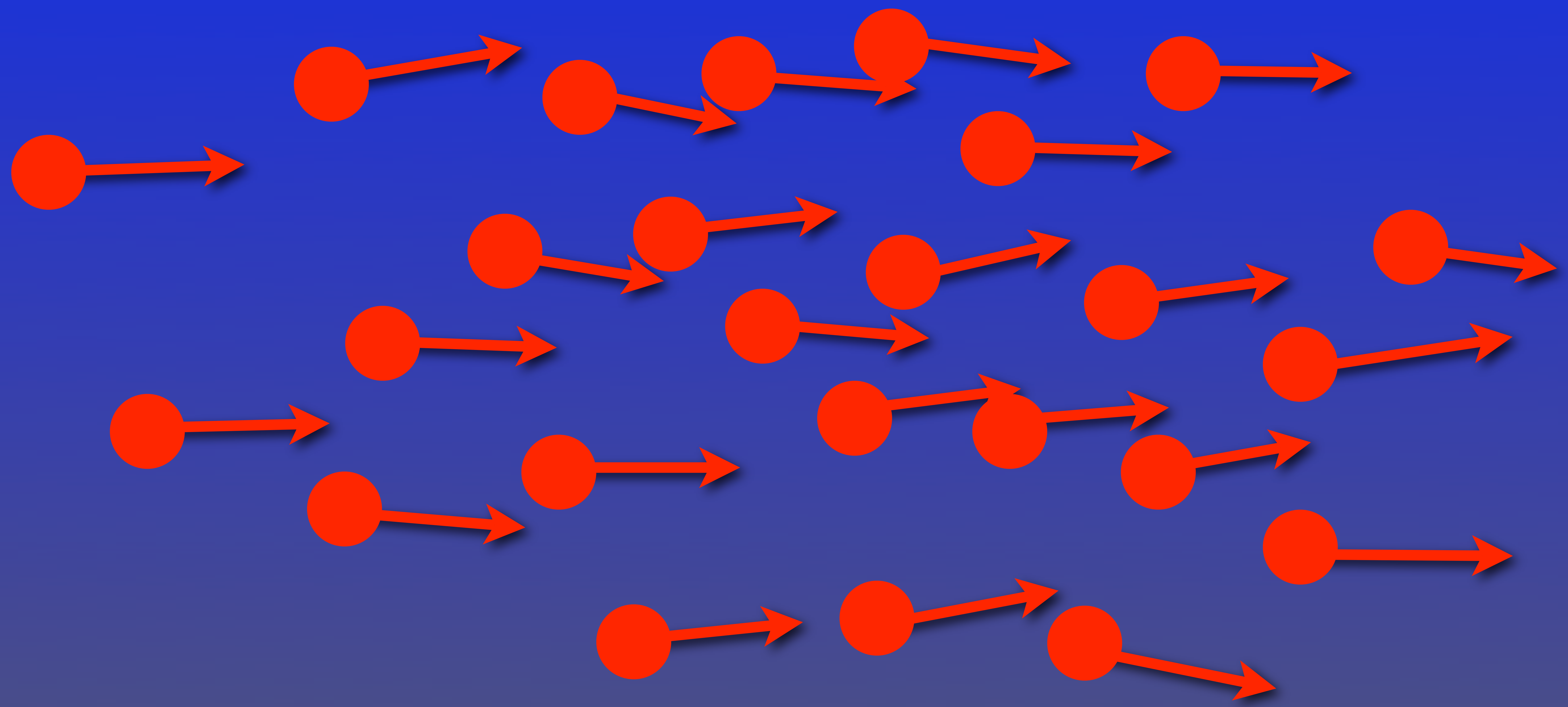
Intra-beam Scattering

Intra-beam Scattering

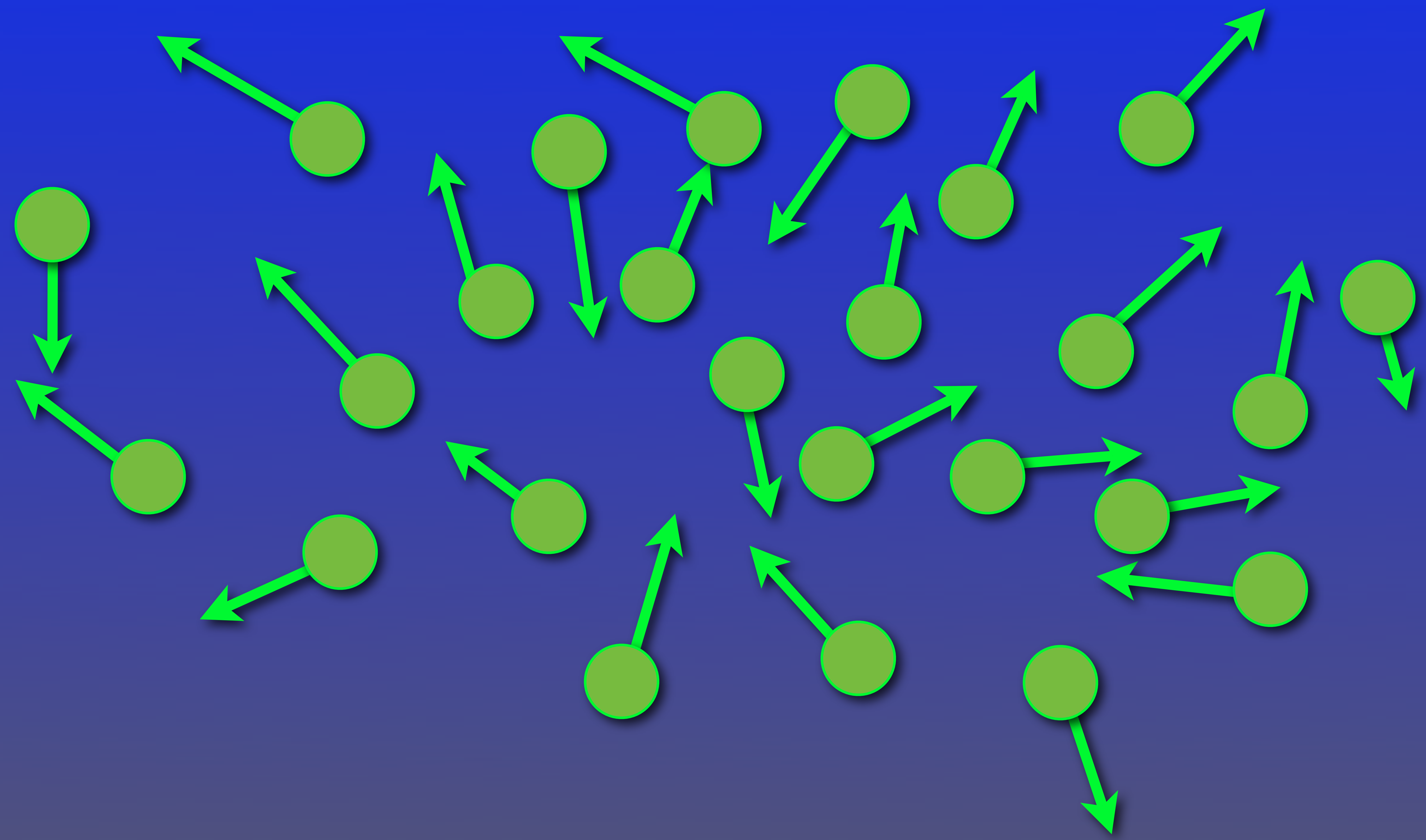
Luminosity

$$\mathcal{L} = \frac{N_e N_h}{4\pi\sqrt{\beta^* \epsilon}} f$$

Intra-beam Scattering



Intra-beam Scattering



Intra-beam Scattering

Dispersion

$$x_\beta = x - D\frac{\Delta p}{p}$$

Coulomb Scatter

Intra-beam Scattering

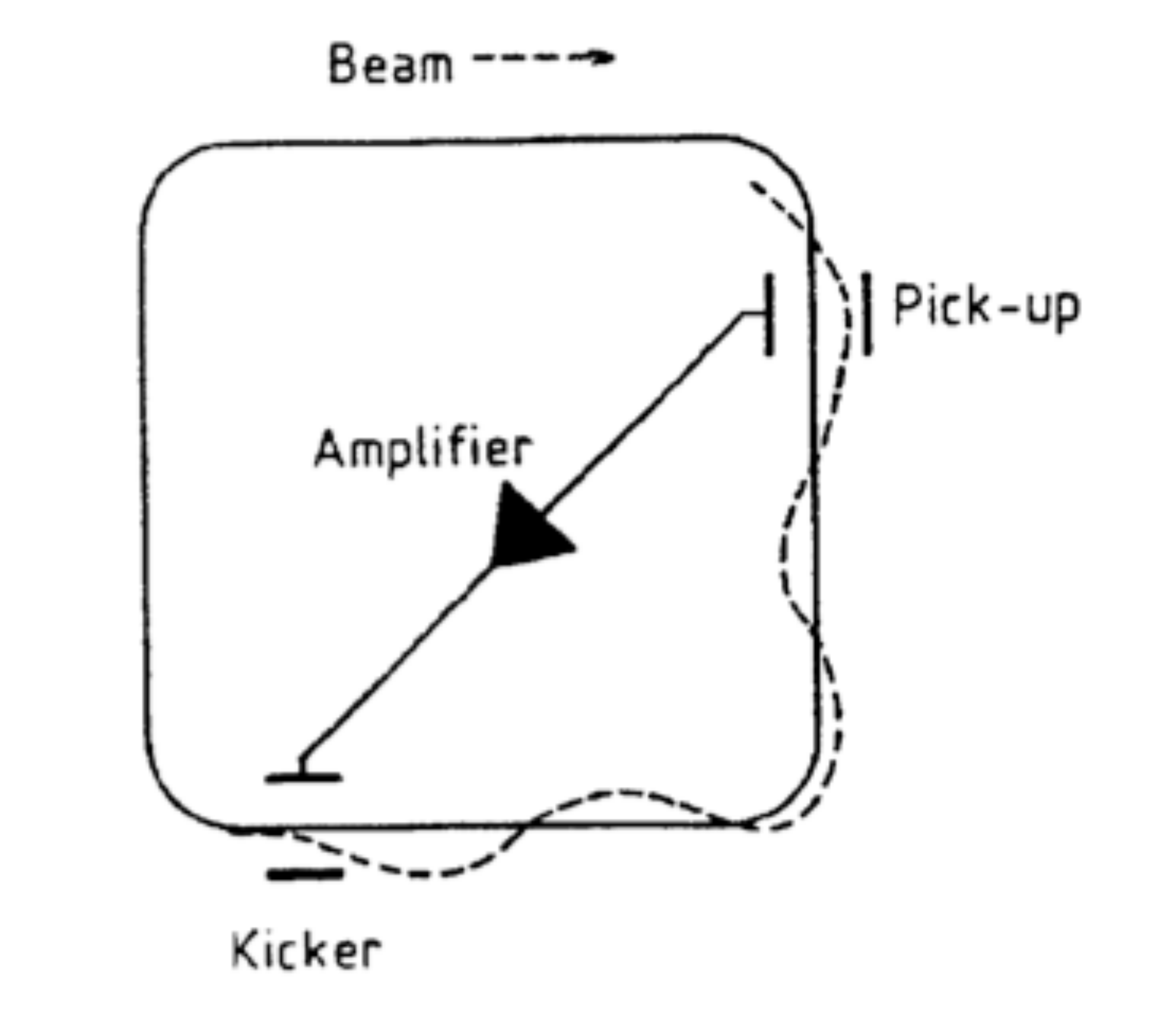
Dispersion

$$x_\beta = x - D\frac{\Delta p}{p}$$

Coulomb Scatter

$$\Delta(\epsilon_{x,1} + \epsilon_{x,2}) = 2\frac{\pi}{\beta_x}\frac{p_x}{p}^2[(D\gamma)^2 - \beta_x^2]$$

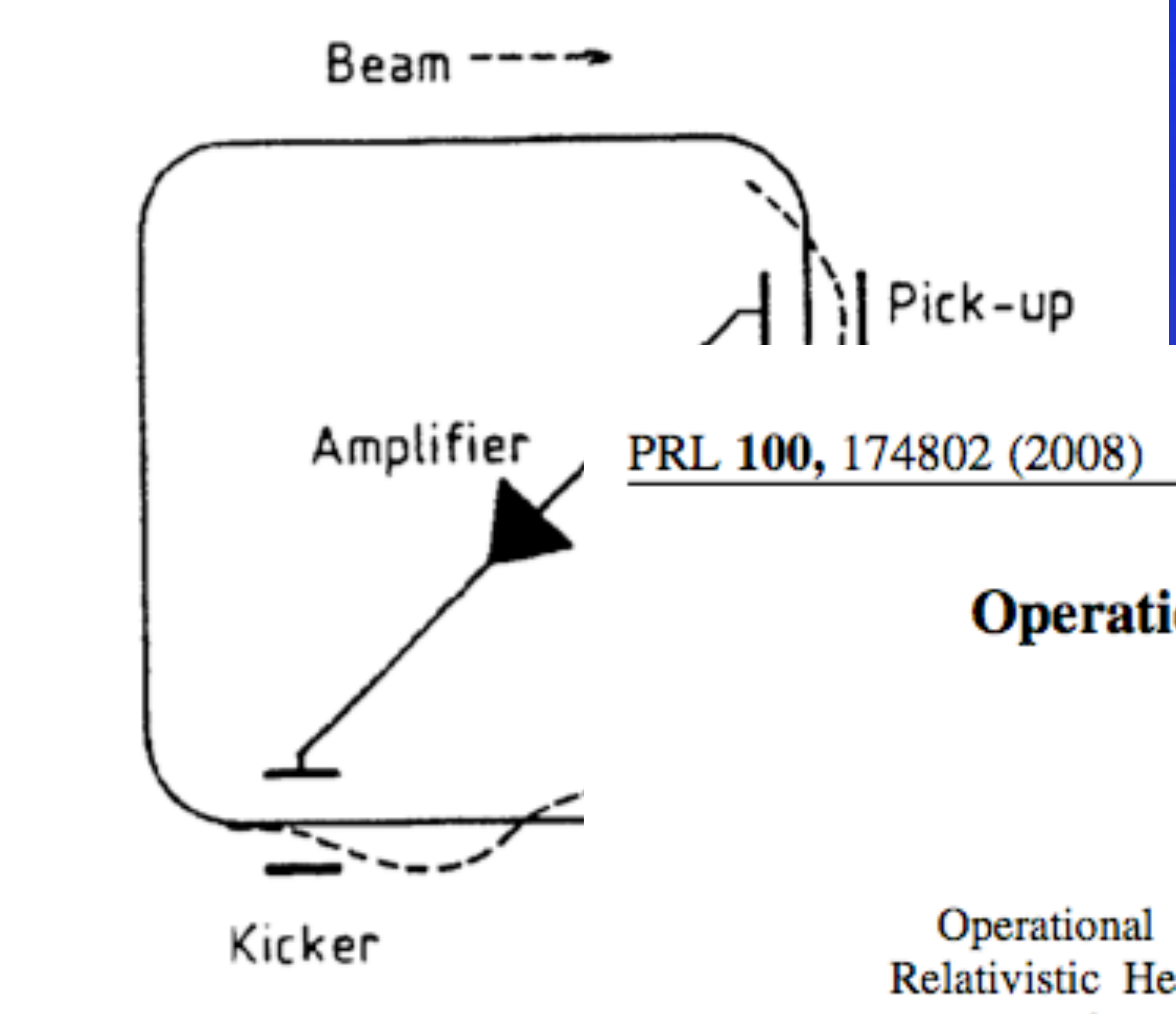
Intra-beam Scattering



$$\frac{1}{\tau} \propto \frac{2W}{N}$$

Shamelessly lifted from D. Mohl "Stochastic Cooling"

Intra-beam Scattering



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PRL **100,** 174802 (2008) PHYSICAL REVIEW LETTERS week ending 2 MAY 2008

Operational Stochastic Cooling in the Relativistic Heavy-Ion Collider

M. Blaskiewicz,* J. M. Brennan, and F. Severino

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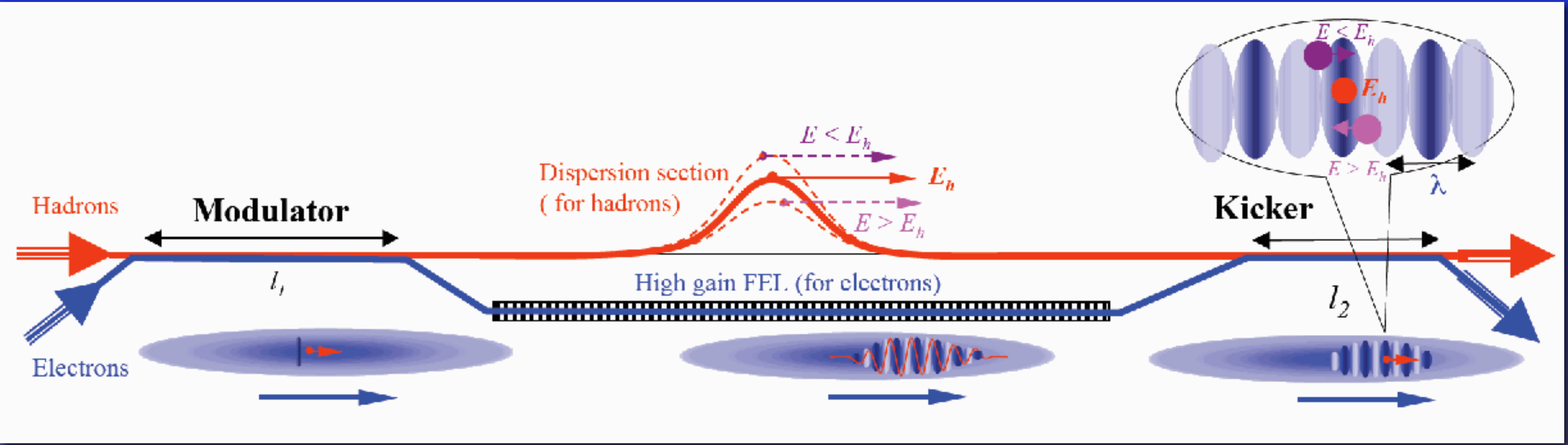
(Received 2 October 2007; published 2 May 2008)

Operational stochastic cooling of 100 GeV/nucleon gold beams has been achieved in the BNL Relativistic Heavy-Ion Collider. We discuss the physics and technology of the longitudinal cooling system and present results with the beams. A simulation algorithm is described and shown to accurately model the system.

Shamelessly lifted from D. Mohl "Stochastic Cooling"

Coherent Electron Cooling

Coherent Electron Cooling



Coherent Electron Cooling

Equations of Motion

$$\epsilon' = -\xi_0 \sin(k_r D_\ell \epsilon + \theta_{FEL}) + V_0 \phi$$

$$\phi' = -\eta\epsilon$$

Coherent Electron Cooling

Equations of Motion

$$\epsilon' = -\xi_0 \sin(k_r D_\ell \epsilon + \theta_{FEL}) + V_0 \phi$$

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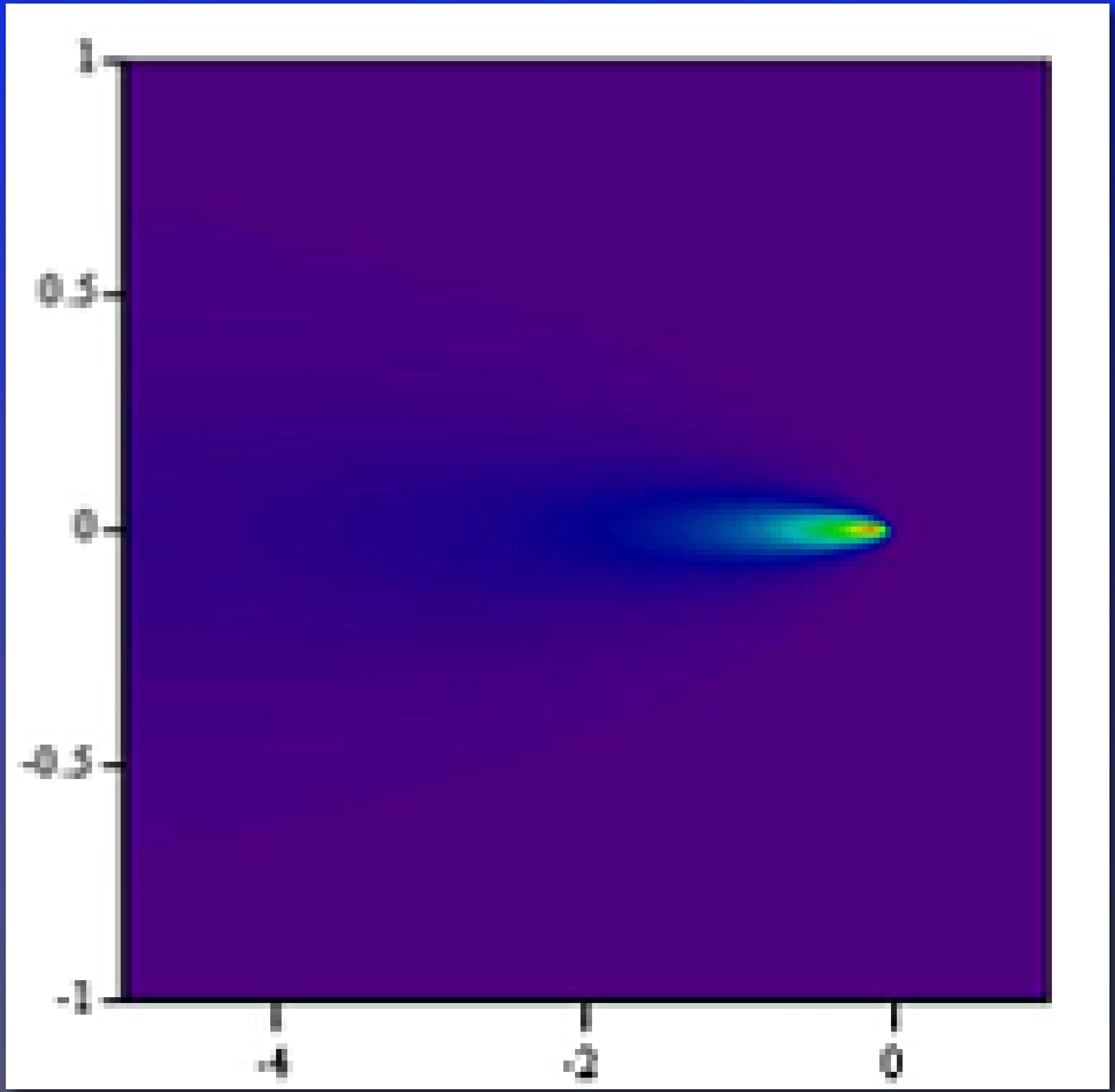
FEL-dependent parameters

Debye Screening



Assumptions:

- Anisotropic velocity distribution
- kappa-2 distribution
- Free, infinite plasma

Lifted from G. Wang and M. Blaskiewicz, PRE (2008)

Debye Screening

Static Screening

$$\tilde{n}_1(\vec{x}, t) = \frac{Z}{4\pi(\sigma_x \sigma_y \sigma_z)} \frac{1}{\bar{r}} e^{-\bar{r}}$$

Dynamic Screening

$$\tilde{n}_1(\vec{x}, t) = \int_0^t dt' \, \frac{Z(\sigma_x \sigma_y \sigma_z)^{-1} \omega_p t' \sin(\omega_p t')}{\pi^2 \left(t'^2 + \sum_\imath \frac{(x_\imath + v_{0\imath} t')^2}{\sigma_\imath^2} \right)^2}$$

Lifted from G. Wang and M. Blaskiewicz, PRE (2008)

Debye Screening

Model Assumptions

- Infinitely wide electron beam
- No external confinement (betatron oscillations)
- Particular energy distribution

Free-electron lasers

JOURNAL OF APPLIED PHYSICS VOLUME 42, NUMBER 5 APRIL 1971

Stimulated Emission of Bremsstrahlung in a Periodic Magnetic Field

JOHN M. J. MADEY
Physics Department, Stanford University, Stanford, California 94305

(Received 20 February 1970; in final form 21 August 1970)

The Weizsäcker–Williams method is used to calculate the gain due to the induced emission of radiation into a single electromagnetic mode parallel to the motion of a relativistic electron through a periodic transverse dc magnetic field. Finite gain is available from the far-infrared through the visible region raising the possibility of continuously tunable amplifiers and oscillators at these frequencies with the further possibility of partially coherent radiation sources in the ultraviolet and x-ray regions to beyond 10 keV. Several numerical examples are considered.

Free-electron lasers

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First Operation of a Free-Electron Laser*

D. A. G. Deacon,† L. R. Elias, J. M. J. Madey, G. J. Ramian, H. A. Schwettman, and T. I. Smith
High Energy Physics Laboratory, Stanford University, Stanford, California 94305
(Received 17 February 1977)

A free-electron laser oscillator has been operated above threshold at a wavelength of 3.4 μm.

Ever since the first maser experiment in 1954, physicists have sought to develop a broadly tunable source of coherent radiation. Several ingenious techniques have been developed, of which the best example is the dye laser. Most of these devices have relied upon an atomic or a molecular active medium, and the wavelength and tuning range has therefore been limited by the details of atomic structure.

Several authors have realized that the constraints associated with atomic structure would not apply to a laser based on stimulated radiation by free electrons.[1-5] Our research has focused on the interaction between radiation and an electron beam in a spatially periodic transverse magnetic field. Of the schemes which have been proposed, this approach appears the best suited to the generation of coherent radiation in the infrared, the visible, and the ultraviolet, and also has the potential for yielding very high average power. We have previously described the results of a measurement of the gain at 10.6 μm.[6] In this Letter we report the first operation of a free-electron laser oscillator.

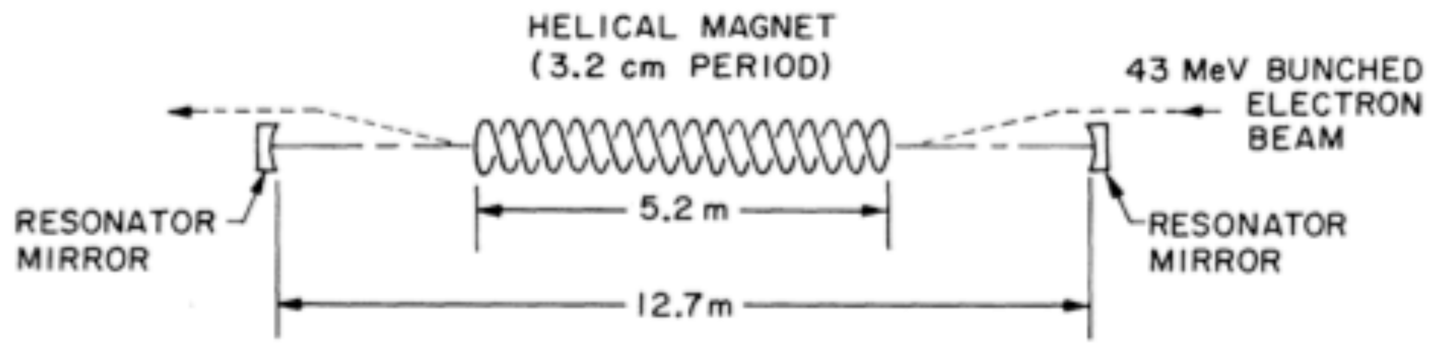

FIG. 1. Schematic diagram of the free-electron laser oscillator. (For more details see Ref. 6.)

Free-electron lasers

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FIRST LASING OF THE *LCLS* X-RAY FEL AT 1.5 Å

P. Emma, for the *LCLS* Commissioning Team; *SLAC*, *Stanford*, CA 94309, USA

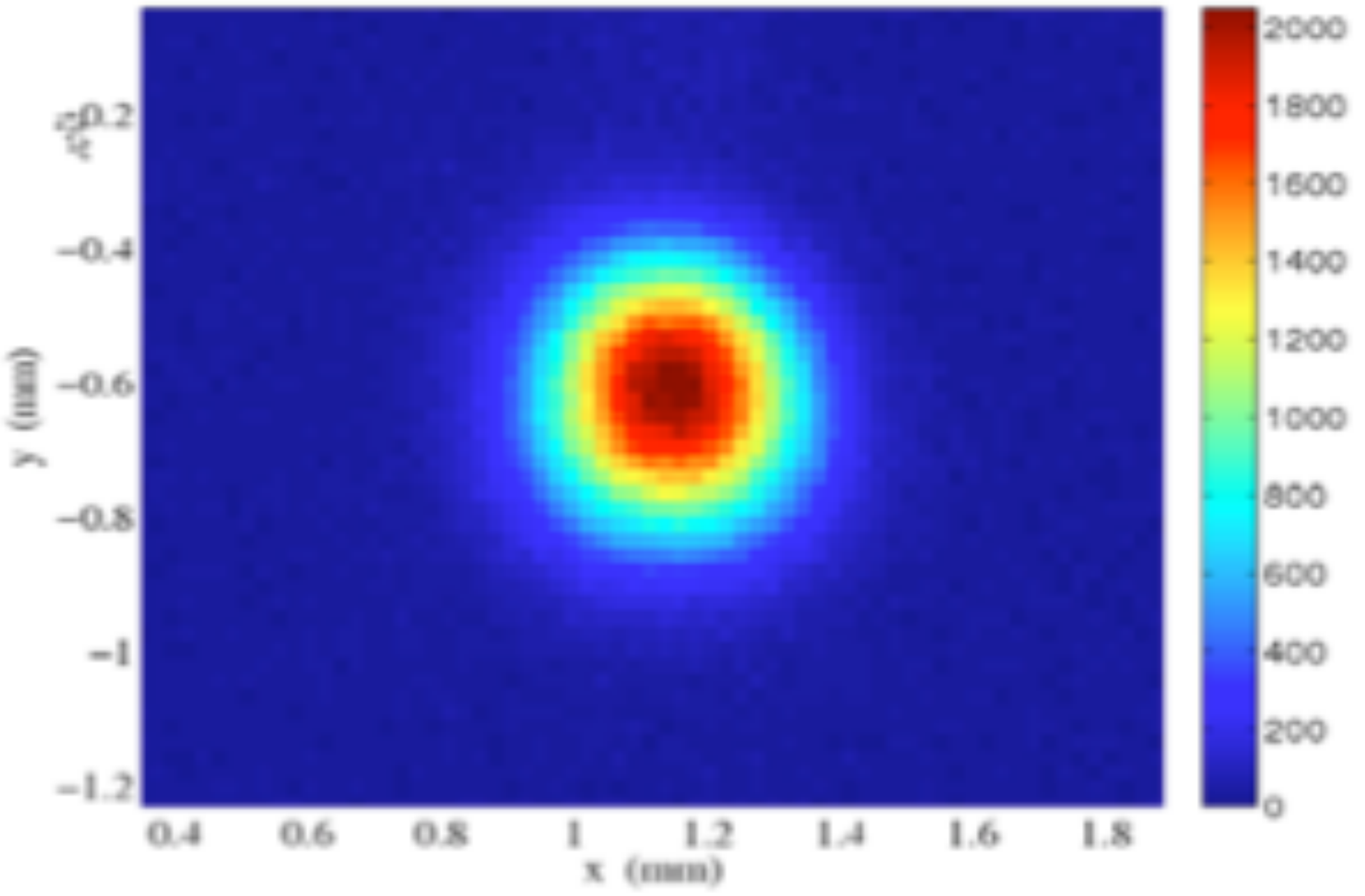

Figure 10: FEL x-rays at 1.5 Å on a YAG screen 50 m after the last inserted undulator (see Table 1 for measured parameters).

First Operation of a Free-Electron Laser*

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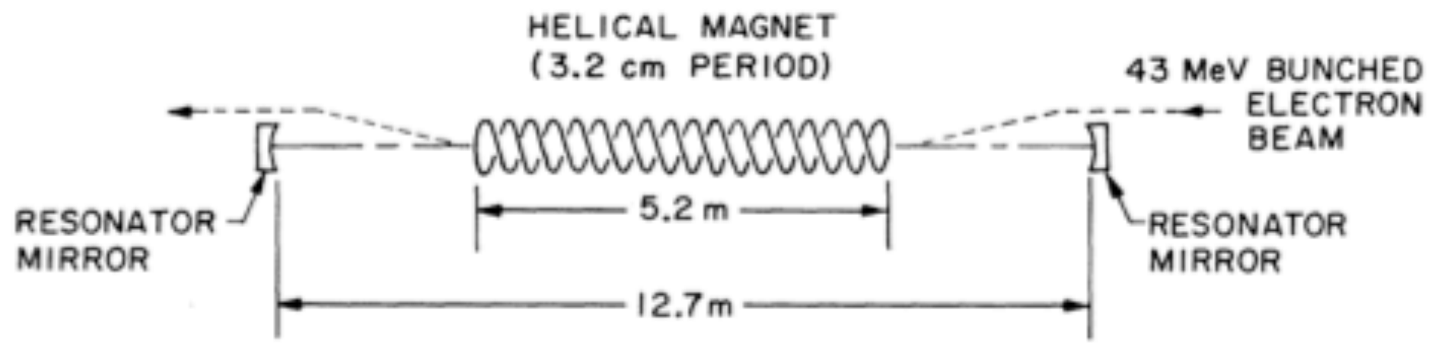
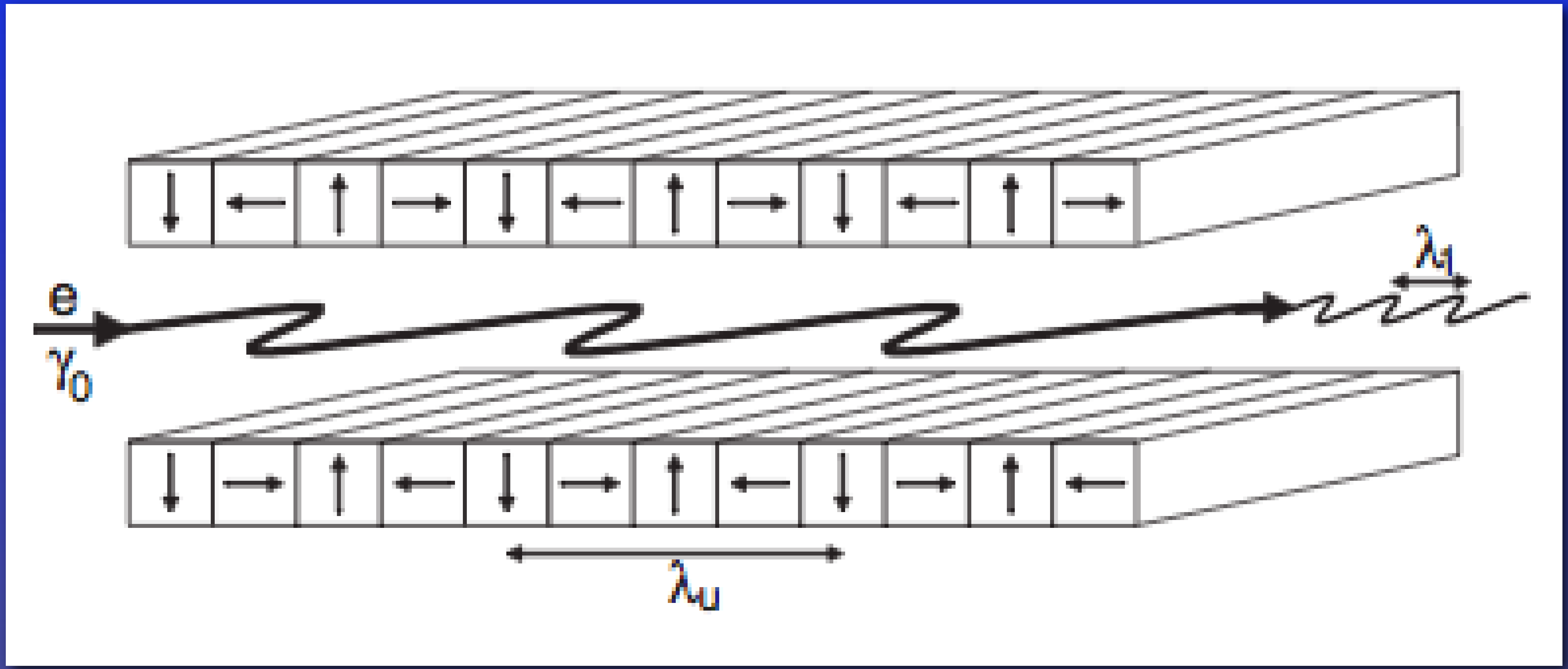


Schematic diagram of the free-electron laser oscillator. (For more details see Ref. 6.)

Free-electron lasers



Lifted from Z. Huang and K.-J. Kim, PRST-AB (2007)

Free-electron lasers

Electron Velocity

$$\vec{v}_\perp(z) \propto e^{\imath k_w z}$$

Electric Field

$$\vec{E} \propto e^{\imath(\omega_r t/c - k_r z)}$$

Free-electron lasers

Electron Velocity

$$\vec{v}_\perp(z) \propto e^{\imath k_w z}$$

Electric Field

$$\vec{E} \propto e^{\imath(\omega_r t/c - k_r z)}$$

Energy Exchange

$$\frac{d\mathcal{E}}{dz} \propto \vec{v}_\perp \cdot \vec{E} \propto e^{\imath(k_w z + \omega_r t/c - k_r z)}$$

Free-electron lasers

Ponderomotive Phase

$$\psi = k_w z + \omega_r t/c - k_r z$$

Resonance Condition

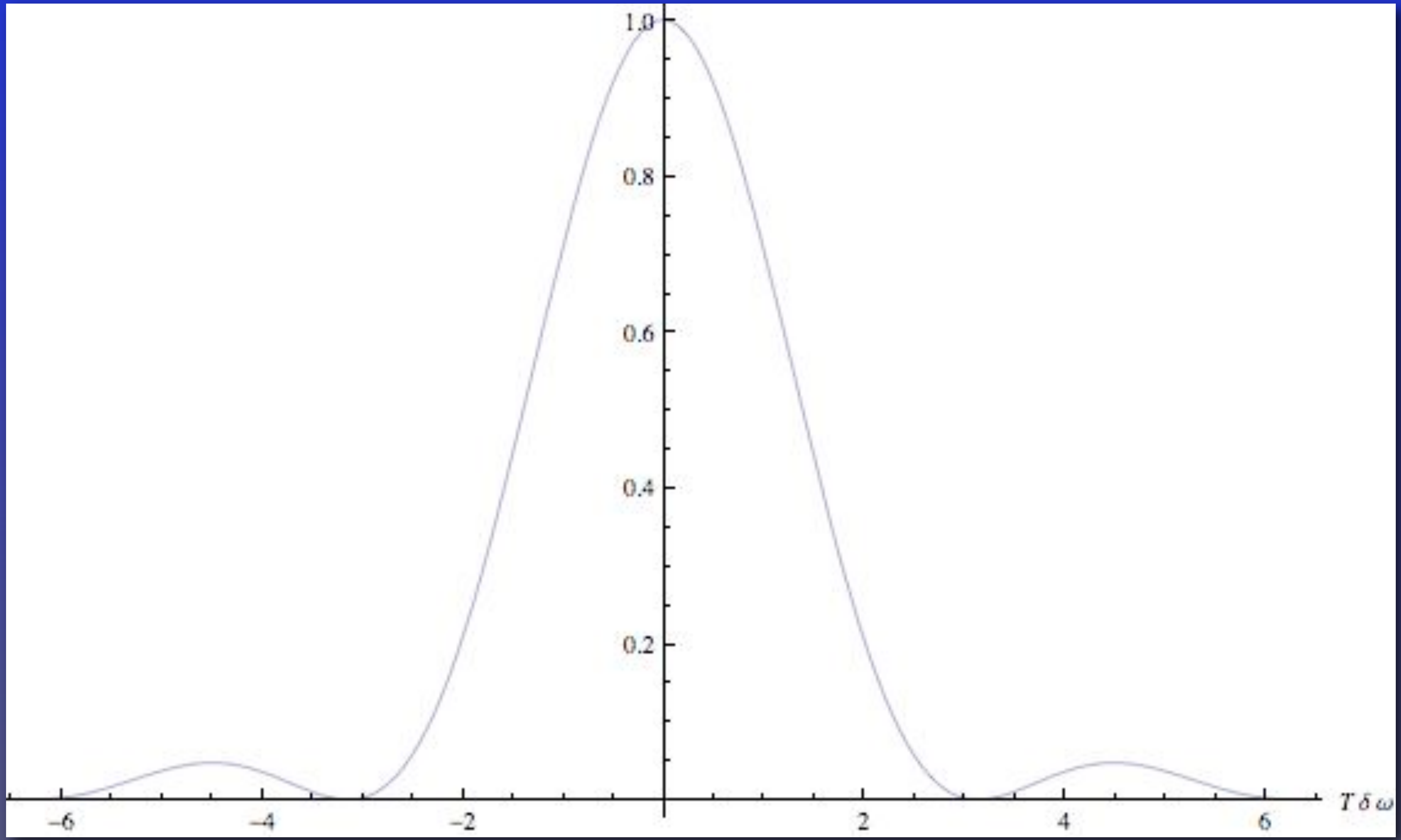
$$\frac{d\psi}{dz} = 0 \rightarrow \lambda_r = \frac{\lambda_w}{2\gamma_0^2}(1 + a_w^2)$$

Free-electron lasers

Bandwidth

$$\frac{1}{T}\int_{-T}^{T} dt\; e^{\imath\omega t}\cos(\omega_0 t) \approx \frac{\sin(\delta\omega\; T)}{\delta\omega\; T}$$

$$\sigma_{FWHM} \approx \sqrt{2}\;\delta\omega\; T$$



Free-electron lasers

Tunable frequency

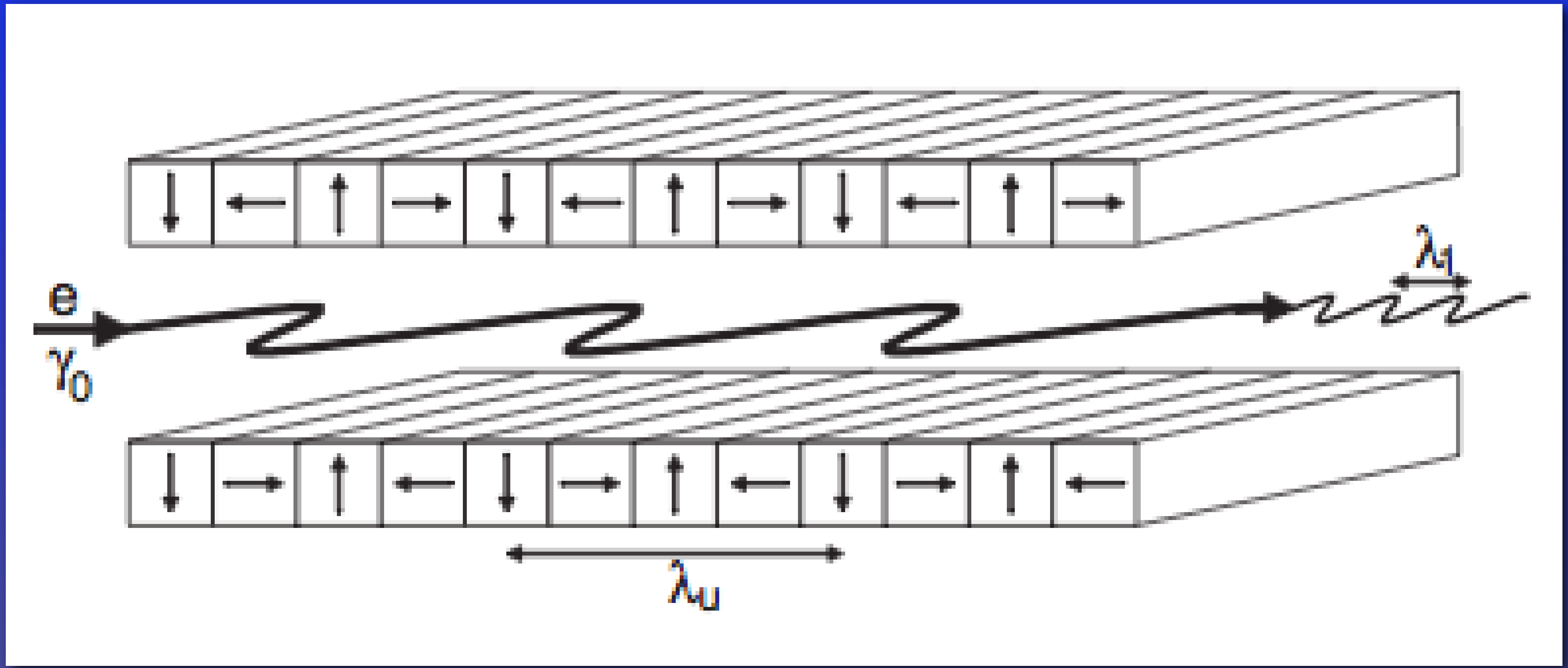
$$\lambda_r \propto \gamma_0^{-2}(1 + a_w^2)$$

Narrow bandwidth

$$\frac{\delta\omega}{\omega_r} \sim \frac{1}{N_w}$$

Three-Dimensional FEL Theory

Three-Dimensional FEL Theory



Lifted from Z. Huang and K.-J. Kim, PRST-AB (2007)

Three-Dimensional FEL Theory

FELs - the basics

Resonance Wavelength

$$\lambda_r = \frac{\lambda_w}{2\gamma_0^2}(1 + K^2)$$

Pierce Parameter

$$\rho = (k_w L_G)^{-1}$$

Gain Length

$$L_G = \left(\frac{\mathcal{E}_0^2 c^2 \gamma_0}{2\pi\nu e^3 K k_w n_0} \right)^{1/3}$$

From Saldin et al. NIM A (1992)

Three-Dimensional FEL Theory

FELs - the basics

Resonance Wavelength

Diffraction Length

$$\ell^2 = \frac{2\nu\omega_r}{L_G c} \sim (1\mathrm{mm})^2$$

Pierce Parame

$$\rho = (k_w L_G)^{-1}$$

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$$L_G = \left(\frac{\mathcal{E}_0^2 c^2 \gamma_0}{2\pi\nu e^3 K k_w n_0} \right)^{1/3}$$

From Saldin et al. NIM A (1992)

Three-Dimensional FEL Theory

Single Particle Equations of Motion

$$\delta\left(\int \mathcal{H}dt - p_z dz\right) = 0$$

Three-Dimensional FEL Theory

Single Particle Equations of Motion

$$\delta\left(\int \mathcal{H}dt - p_z dz\right) = 0$$

$$p_z \approx \frac{\mathcal{E}_0 + \mathcal{E}}{c} - \frac{1}{2}\frac{1}{\mathcal{E}_0 c}\left(1 - \frac{\mathcal{E}}{\mathcal{E}_0} + \left(\frac{\mathcal{E}}{\mathcal{E}_0}\right)^2\right) \times$$

$$\left\{\frac{e^2}{c^2}(\vec{A}_w^2 + 2\vec{A}_w \cdot \vec{A}_l) + m^2c^2\right\} + \frac{e}{c}A_z$$

Three-Dimensional FEL Theory

Single Particle Equations of Motion

$$\frac{d\mathcal{E}}{dz} = \frac{1}{\mathcal{E}_0 c}\left(1 - \frac{\mathcal{E}}{\mathcal{E}_0}\right)\frac{e^2}{c^2}\vec{A}_w \cdot \frac{\partial \vec{A}_l}{\partial t} - \frac{e}{c}\frac{\partial A_z}{\partial t}$$

$$\frac{dt}{dz} = \frac{1}{c} - \frac{1}{2}\left(-\frac{1}{\mathcal{E}_0} + 2\frac{\mathcal{E}}{\mathcal{E}_0^2}\right)\left\{\left(\frac{e}{c}\vec{A}_w\right)^2 + m^2c^2 + 2\frac{e^2}{c^2}\vec{A}_w \cdot \vec{A}_l\right\}\frac{1}{\mathcal{E}_0 c}$$

Three-Dimensional FEL Theory

Maxwell Equations

$$\frac{1}{(\sqrt{2\pi})^3}\int d\nu\ d^2k_\perp\ e^{\imath\nu\omega_r(z/c-t)}e^{\imath\vec{k}_\perp\cdot\vec{r}_\perp}\left(2\imath\nu\omega_r/c\ \partial_z\tilde{A}_l - k_\perp^2\tilde{A}_l\right) = \frac{4\pi}{c}\vec{j}_\perp$$

$$\partial_t E_z = -\frac{4\pi}{c}j_z$$

Three-Dimensional FEL Theory

Maxwell Equations

$$\vec{A}_w \cdot \tilde{A}_l = e^{-\imath \frac{c k_\perp^2}{2\nu\omega_r} z} e^{\imath k_w z} \left\{ \vec{A}_w \cdot \tilde{A}_l \,|_{z=0} + \frac{\imath\pi}{\nu\omega_r} \frac{K}{\gamma_0} A_w \int_0^z \tilde{j}_z dz' \right\}$$

$$\tilde{E}_z = -\frac{4\pi\imath}{c\nu\omega_r} \tilde{j}_z$$

Three-Dimensional FEL Theory

Maxwell-Vlasov Formalism

$$\frac{df}{dz} = \frac{\partial f}{\partial z} + t' \frac{\partial f}{\partial t} + \mathcal{E}' \frac{\partial f}{\partial \mathcal{E}} = 0$$

Conservation of phase space density

Three-Dimensional FEL Theory

Maxwell-Vlasov Formalism

$$\frac{df}{dz} = \frac{\partial f}{\partial z} + t' \frac{\partial f}{\partial t} + \mathcal{E}' \frac{\partial f}{\partial \mathcal{E}} = 0$$

Conservation of phase space density

Assume $\quad f = f_0 + f_1 \qquad |f_1| \ll |f_0|$

$$\vec{A} \propto \int f_1 d\mathcal{E}$$

Three-Dimensional FEL Theory

Maxwell-Vlasov Formalism

$$\frac{df}{dz} = \frac{\partial f}{\partial z} + t'\frac{\partial f}{\partial t} + \mathcal{E}'\frac{\partial f}{\partial \mathcal{E}} = 0$$

Conservation of phase space density

Assume $\quad f = f_0 + f_1 \qquad |f_1| \ll |f_0|$

$$\vec{A} \propto \int f_1 d\mathcal{E}$$

$$\frac{\partial f_1}{\partial z} + t'_{\mathcal{O}((f_1)^0)}\frac{\partial f_1}{\partial t} + \mathcal{E}'_{\mathcal{O}((f_1)^1)}\frac{\partial f_0}{\partial \mathcal{E}} + \mathcal{O}(f_1^2) = 0$$

Three-Dimensional FEL Theory

Normalized Coordinates

Detuning

$$\hat{C} = \frac{\omega_r - \omega}{\omega_r \rho}$$

Energy Deviation

$$\hat{P} = \frac{2\omega \mathcal{E}}{\omega_r \rho \mathcal{E}_0}$$

Diffraction

$$\hat{k}_\perp^2 = k_\perp^2 \frac{c L_g}{2\omega_r}$$

Three-Dimensional FEL Theory

Maxwell-Vlasov Equation of Motion

$$\tilde{j}_z = -ec\frac{\rho\mathcal{E}_0}{2\nu}\int d\hat{P}\ e^{\imath(\hat{C}+\hat{P}-\hat{k}_\perp^2)\hat{z}}\,\tilde{f}_1\ |_{\hat{z}=0} +$$

$$\int d\hat{P}\ \int_0^{\hat{z}} d\hat{z}' e^{\imath(\hat{C}+\hat{P}-\hat{k}_\perp^2)(\hat{z}'-\hat{z})}\int d^2\hat{q}\ e^{-\imath(\hat{q}^2-\hat{k}_\perp^2)\hat{z}'}\times$$

$$\left\{\hat{\mathcal{U}}_0 + \int_0^{\hat{z}'} d\hat{z}''\ \tilde{j}_z + \imath\hat{\Lambda}_p^2\tilde{j}_z\right\}\frac{d\hat{F}}{d\hat{P}}\hat{G}(\vec{q}-\vec{k}_\perp)$$

Three-Dimensional FEL Theory

Maxwell-Vlasov Equation of Motion

$$\mathcal{L}[f(x)] = F(s) = \int_0^\infty ds\; e^{-sx} f(x)$$

Solves initial value problem as algebraic problem

Separates out the integral equation

Poles in s determine dispersion relation from Cauchy integral formula

Three-Dimensional FEL Theory

Dispersion Relation

$$s = \frac{\hat{D}}{1 - \imath \hat{\Lambda}_p^2 \hat{D}}$$

$$\hat{D}(s) = \int d\hat{P}\, \frac{d\hat{F}}{d\hat{P}} \frac{1}{s + \imath(\hat{C} + \hat{P})}$$

From Saldin et al. NIM A (1992)

Three-Dimensional FEL Theory

Infinite Beam

$$\hat{G}(\vec{q}-\vec{k}_\perp) = \delta(\vec{q}-\vec{k}_\perp) \qquad \hat{C}_{3D} = \hat{C} - \hat{k}_\perp^2$$

$$\tilde{j}_z = -ec\frac{\rho\mathcal{E}_0}{2\nu}\sum_\jmath \int d\hat{P}\, \frac{s_\jmath e^{s_\jmath \hat{z}}}{1-\hat{D}'_\jmath + \imath\hat{\Lambda}_p^2\left(\hat{D}_j + s_\jmath \hat{D}'_\jmath\right)} \frac{1}{s_\jmath + \imath(\hat{C}+\hat{P}-\hat{k}_\perp^2)}\tilde{f}_1\,|_{\hat{z}=0}$$

Three-Dimensional FEL Theory

Finite Beam

$$\tilde{j}_z = -ec\frac{\rho\mathcal{E}_0}{2\nu}\int d\hat{P}\; e^{\imath(\hat{C}+\hat{P}-\hat{k}_\perp^2)\hat{z}}\,\tilde{f}_1\,|_{\hat{z}=0} +$$

$$\int d\hat{P}\;\int_0^{\hat{z}} d\hat{z}' e^{\imath(\hat{C}+\hat{P}-\hat{k}_\perp^2)(\hat{z}'-\hat{z})}\int d^2\hat{q}\; e^{-\imath(\hat{q}^2-\hat{k}_\perp^2)\hat{z}'}\times$$

$$\left\{\hat{\mathcal{U}}_0 + \int_0^{\hat{z}'} d\hat{z}''\;\tilde{j}_z + \imath\hat{\Lambda}_p^2\tilde{j}_z\right\}\frac{d\hat{F}}{d\hat{P}}\hat{G}(\vec{q}-\vec{k}_\perp)$$

Three-Dimensional FEL Theory

Finite Beam

$$\tilde{j}_z = -ec\frac{\rho\mathcal{E}_0}{2\nu}\int d\hat{P}\; e^{\imath(\hat{C}+\hat{P}-\hat{k}_\perp^2)\hat{z}}\,\tilde{f}_1\,|_{\hat{z}=0} +$$

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$$\left\{\hat{\mathcal{U}}_0+\int_0^{\hat{z}'} d\hat{z}''\;\tilde{j}_z+\imath\hat{\Lambda}_p^2\tilde{j}_z\right\}\frac{d\hat{F}}{d\hat{P}}\boxed{\hat{G}(\vec{q}-\vec{k}_\perp)}$$

Must account for beam size effects

Three-Dimensional FEL Theory

Finite Beam

Eigenmode Expansion

$$\psi_\ell(\vec{k}_\perp) = \frac{1}{\omega_\ell} \int d^2 q_\perp \; \hat{G}(\vec{k}_\perp - \vec{q}_\perp)\psi_\ell(\vec{q}_\perp)$$

$$\int_0^{\hat{z}} d\hat{z}' \; \tilde{j}_z(\hat{z}') = \sum_\ell \psi_\ell(\vec{k}_\perp) e^{\imath \vec{k}_\perp^2 \hat{z}} a_\ell(\hat{z})$$

Three-Dimensional FEL Theory

Finite Beam

Eigenmode Expansion

$$\psi_\ell(\vec{k}_\perp) = \frac{1}{\omega_\ell} \int d^2 q_\perp \; \hat{G}(\vec{k}_\perp - \vec{q}_\perp) \psi_\ell(\vec{q}_\perp)$$

Optical Guiding

$$\int_0^{\hat{z}} d\hat{z}' \; \tilde{j}_z(\hat{z}') = \sum_\ell \psi_\ell(\vec{k}_\perp) e^{\imath \vec{k}_\perp^2 \hat{z}} a_\ell(\hat{z})$$

Three-Dimensional FEL Theory

Finite Beam

$$j_z = \frac{1}{(\sqrt{2\pi})^3} \int d\nu \; d^2k_\perp e^{\imath k_w z + \imath\nu\omega_r(z/c-t)} e^{\imath \vec{k}_\perp \cdot \vec{r}_\perp} e^{-\imath c k_\perp^2/(2\nu\omega_r) z} \tilde{j}_z$$

Diffraction cancels

$$\int_0^{\hat{z}} d\hat{z}' \; \tilde{j}_z(\hat{z}') = \sum_\ell \psi_\ell(\vec{k}_\perp) e^{\imath \vec{k}_\perp^2 \hat{z}} a_\ell(\hat{z})$$

Three-Dimensional FEL Theory

Finite Beam

Eigenmode Expansion

$$a'_\ell - \imath Q_{m,\ell} a_m = -ec\frac{\rho \mathcal{E}_0}{2\nu} \int d\hat{\mathcal{E}} \int d^2\hat{k}_\perp \; e^{\imath(\hat{C}+\hat{\mathcal{E}}-\hat{k}_\perp^2)\hat{z}} \tilde{f}_1 \,|_0\, \psi_\ell(\hat{k}_\perp) -$$

$$\int d\hat{\mathcal{E}} \int_0^{\hat{z}} d\hat{z}' \; e^{\imath(\hat{C}+\hat{\mathcal{E}})(\hat{z}'-\hat{z})} \times \frac{1}{\omega_\ell} \left\{ a_n + \imath \hat{\Lambda}_p^2 \left[a'_\ell + \imath Q_{m,\ell} a_m \right] \right\} \frac{d\hat{F}}{d\hat{\mathcal{E}}}$$

$$Q_{m,\ell} = \int d^2 k_\perp \; k_\perp^2 \psi_m(\vec{k}_\perp) \psi_\ell(\vec{k}_\perp)$$

Three-Dimensional FEL Theory

Finite Beam

Laplace Transform

$$\left[\left(s - \hat{D}\omega_m(1 + \imath s\hat{\Lambda}_p^2)\right)\delta_{\ell,m} + (1 + \imath\hat{\Lambda}_p^2\omega_m)Q_{\ell,m}\right]a_m = \tilde{f}_1^\ell$$

Three-Dimensional FEL Theory

Finite Beam

One-Dimensional Limit

$$\omega_m = 1$$
$$Q_{\ell,m} = 0$$

Scaling Laws

$$\omega \sim \hat{L}^0$$
$$Q \sim \hat{L}^{-4}$$

$$\psi_\ell(\vec{k}_\perp) = \frac{1}{\omega_\ell} \int d^2q_\perp \; \tilde{G}(\vec{k}_\perp - \vec{q}_\perp)\psi_\ell(\vec{q}_\perp) \rightarrow \Big(\psi_\ell(\vec{k}_\perp) = \psi_\ell(\vec{k}_\perp), \; \omega_\ell = 1\Big)$$

FEL Dispersion Relation

FEL Dispersion Relation

Dispersion Relation

$$s = \frac{\hat{D}}{1 - \imath\hat{\Lambda}_p^2\hat{D}}$$

$$\hat{D}(s) = \int d\hat{P}\,\frac{d\hat{F}}{d\hat{P}}\frac{1}{s + \imath(\hat{C} + \hat{P})}$$

FEL Dispersion Relation

Dispersion Relation

Gaussian through Bell Curves

$$\frac{\Gamma(N)}{\sqrt{2\pi N\sigma^2}\Gamma(N-1/2)}\frac{1}{\left(1+\hat{P}^2/(2\sigma^2 N)\right)^N}$$

FEL Dispersion Relation

Dispersion Relation

From Saldin

$$\hat{D}_{cold}(s) = \frac{\imath}{(s+\imath\hat{C})^2}$$

$$\hat{D}_{\kappa-1}(s) = \frac{\imath}{(s+\hat{q}+\imath\hat{C})^2}$$

From Webb et al. FEL'10

$$\hat{D}_N = \imath\frac{\Gamma[N]}{q_N\Gamma[N-1/2]}\times$$

$$\frac{2\pi}{(N-1)!}\frac{1}{2^{2N-1}}\sum_{m=0}^{N-1}\left(\begin{array}{c} N-1 \\ m \end{array}\right)$$

$$\left\{\frac{2^m m!}{(s+q_N+\imath\hat{C})^{2+m}}q_N^{N-1-m}\frac{(2N-1-m)!}{(N-1)!}\right\}$$

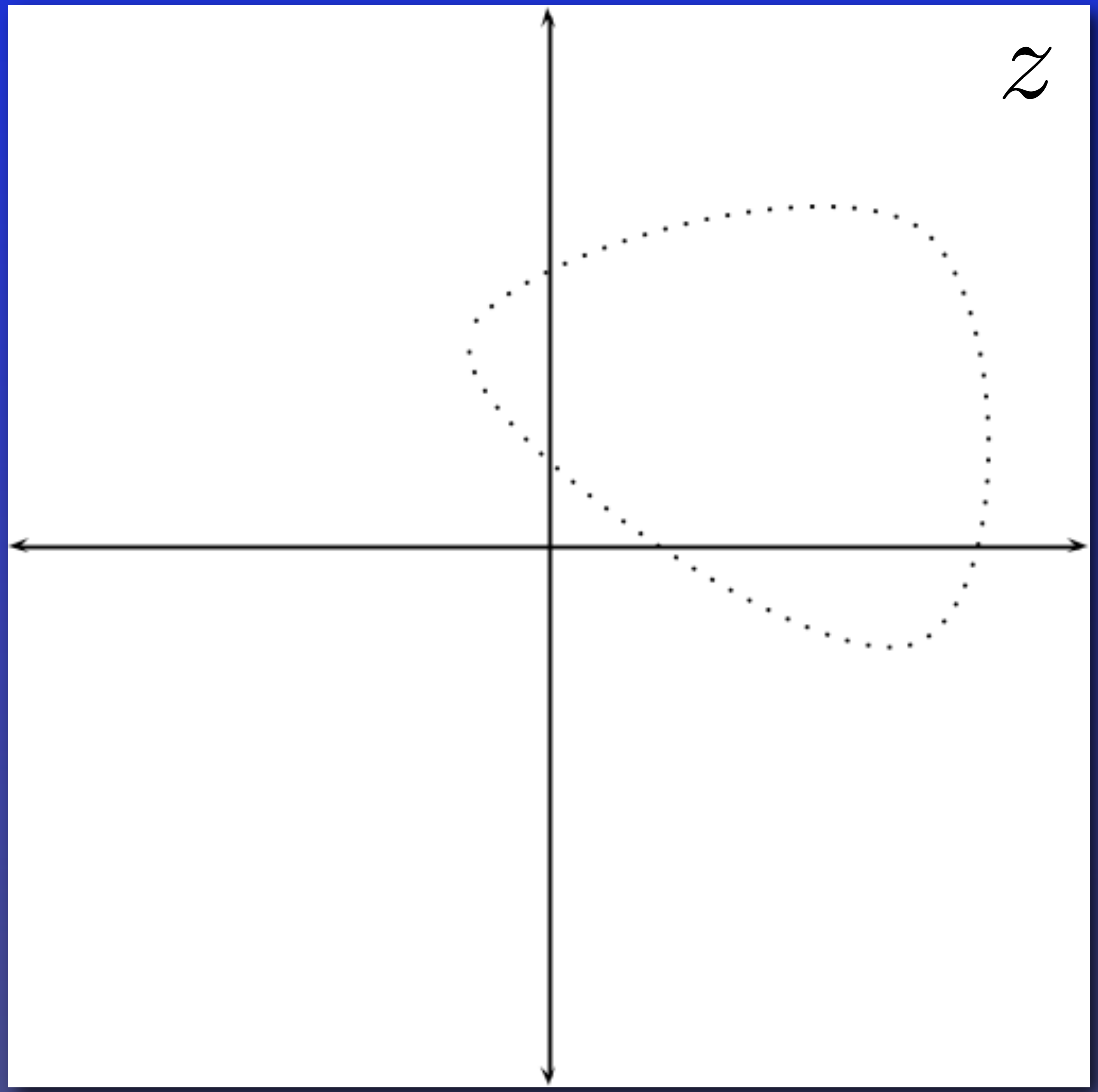
FEL Dispersion Relation

Dispersion Relation

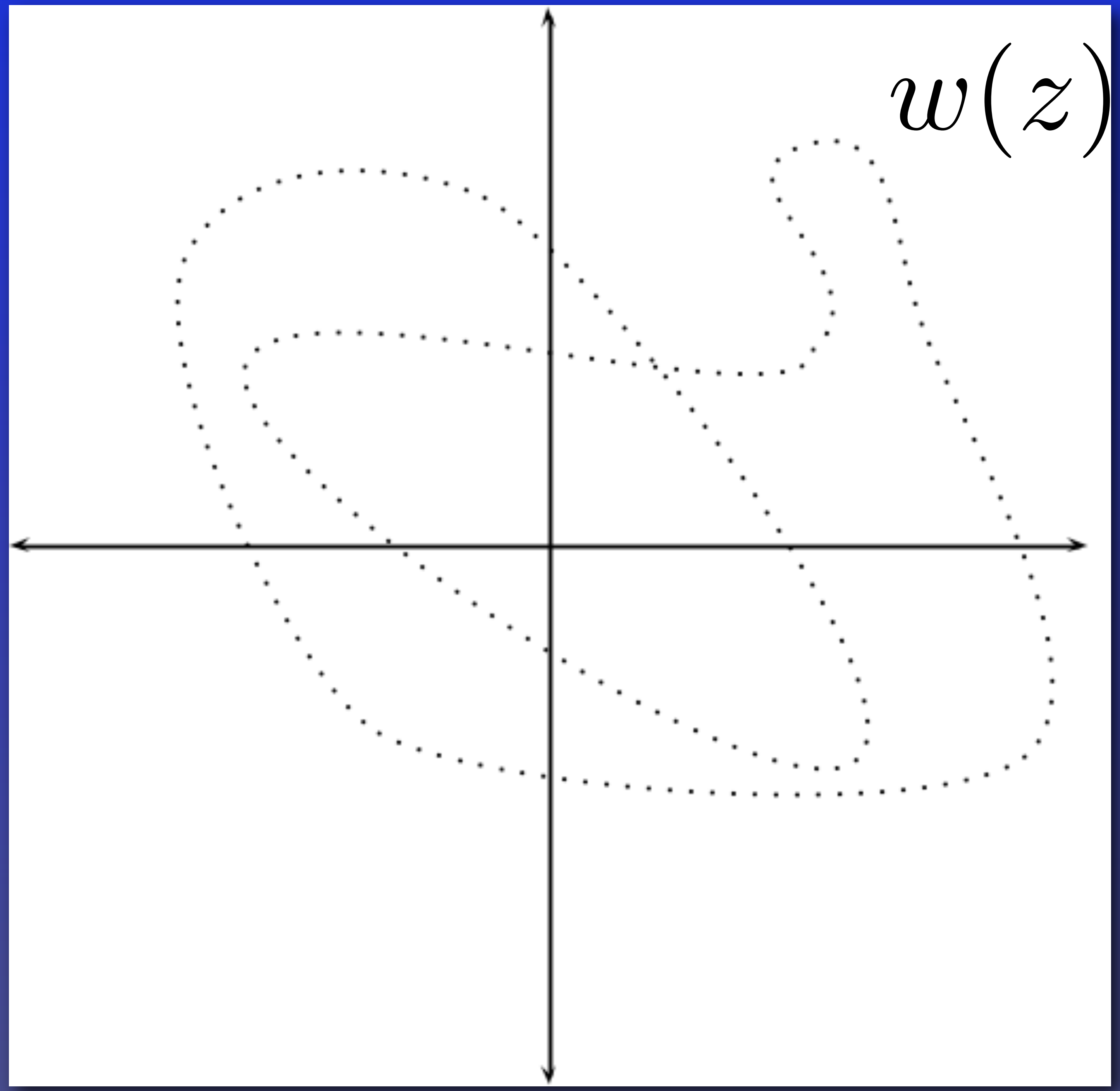
3 roots for cold beam
3 roots of Lorentzian
....
N+2 roots for N-Lorentzian
....
Infinite roots for Gaussian!

FEL Dispersion Relation

Argument Principle



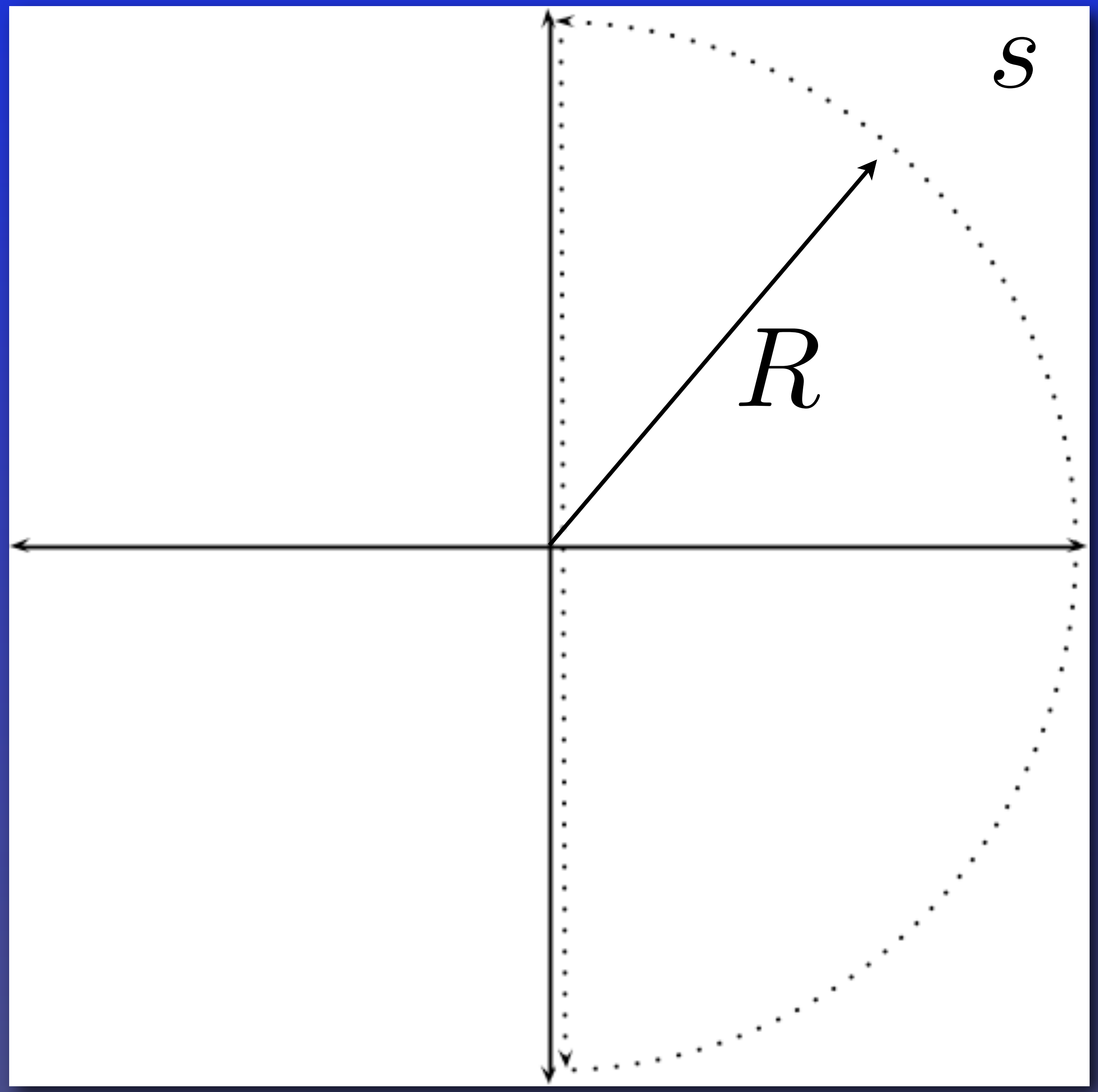
$$\xrightarrow{w(z)}$$



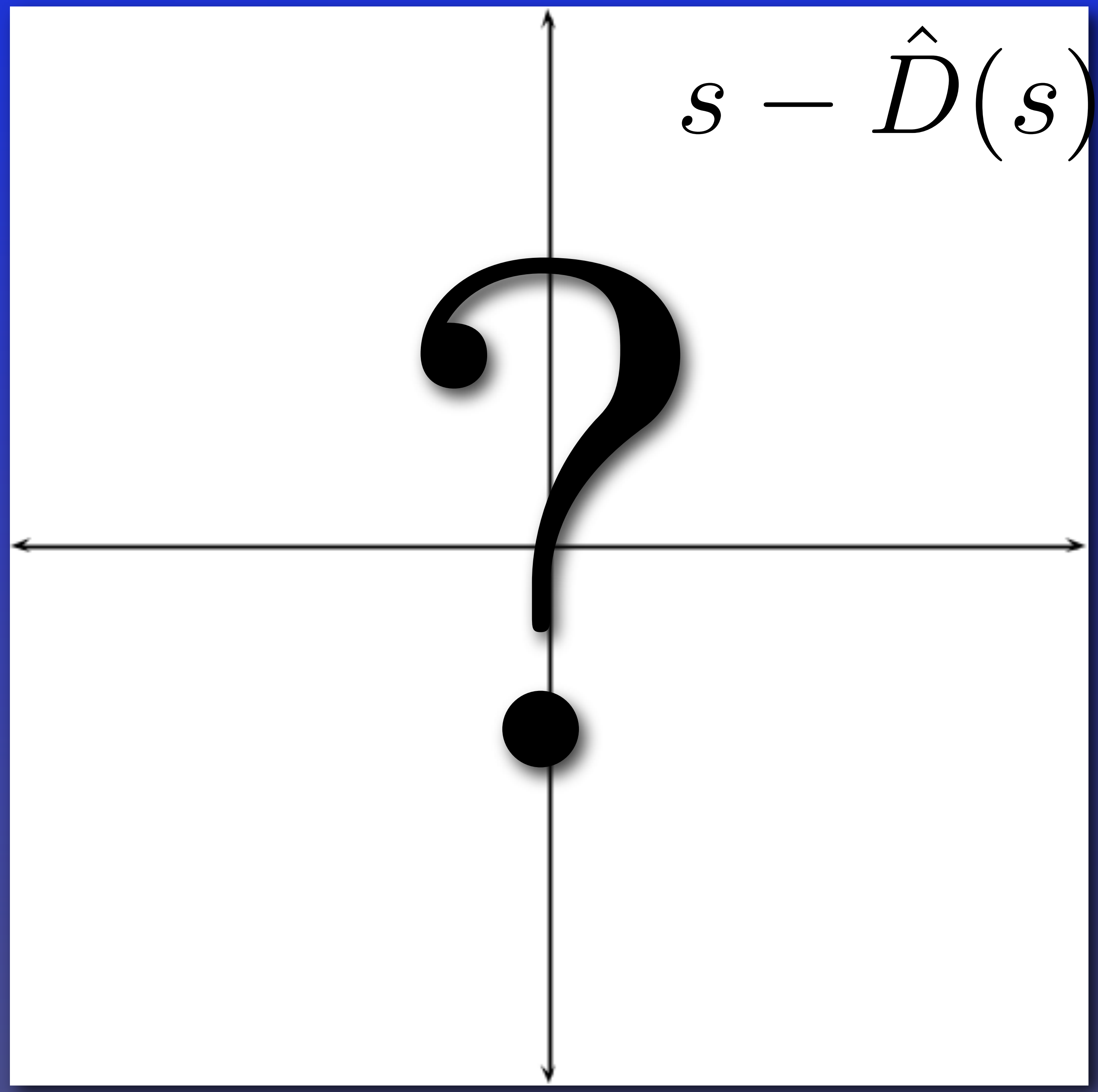
$$Z - P = \frac{1}{2\pi} \oint d(Arg(w(z)))$$

FEL Dispersion Relation

Argument Principle



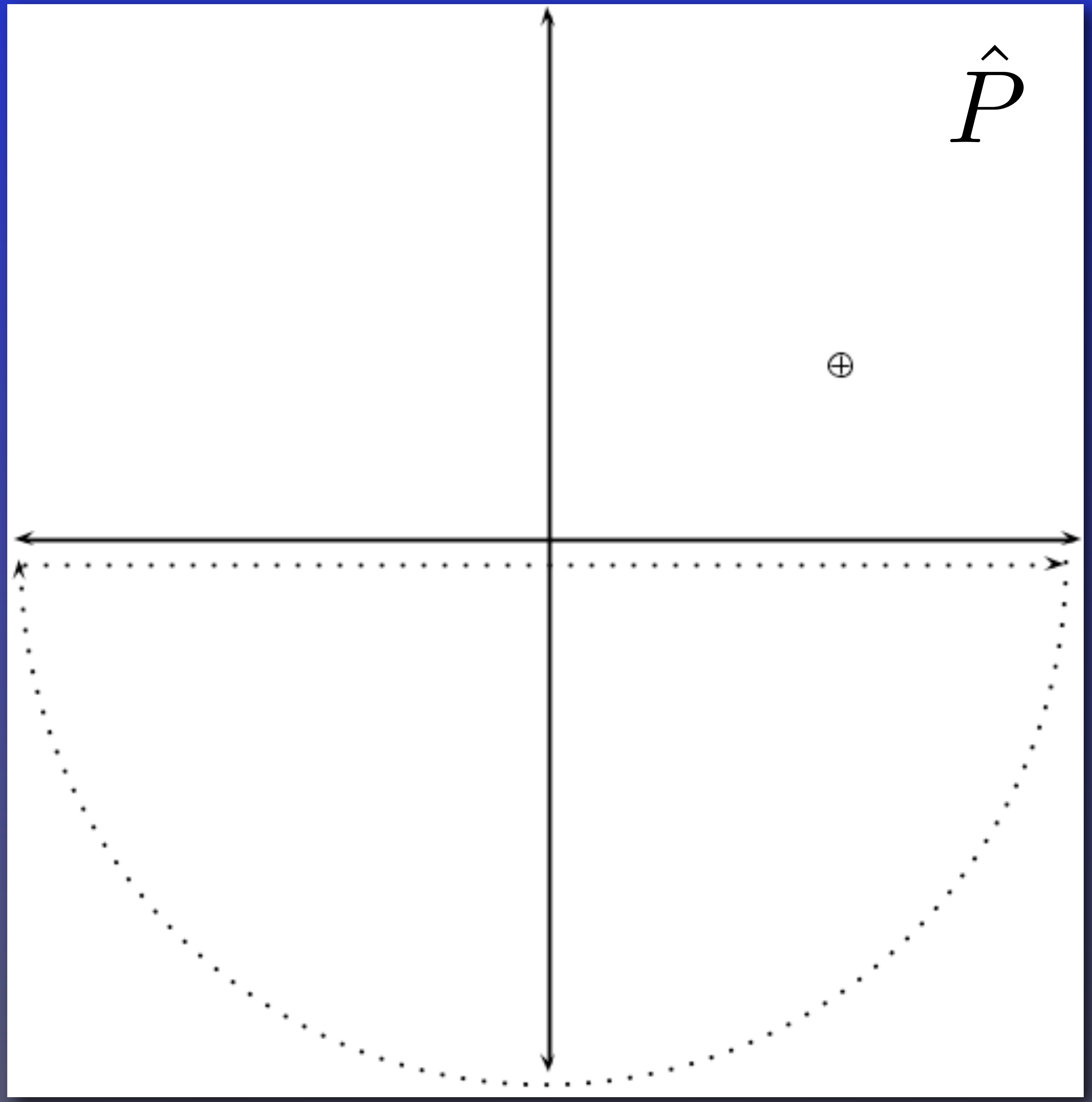
$$\xrightarrow{s - \hat{D}(s)}$$



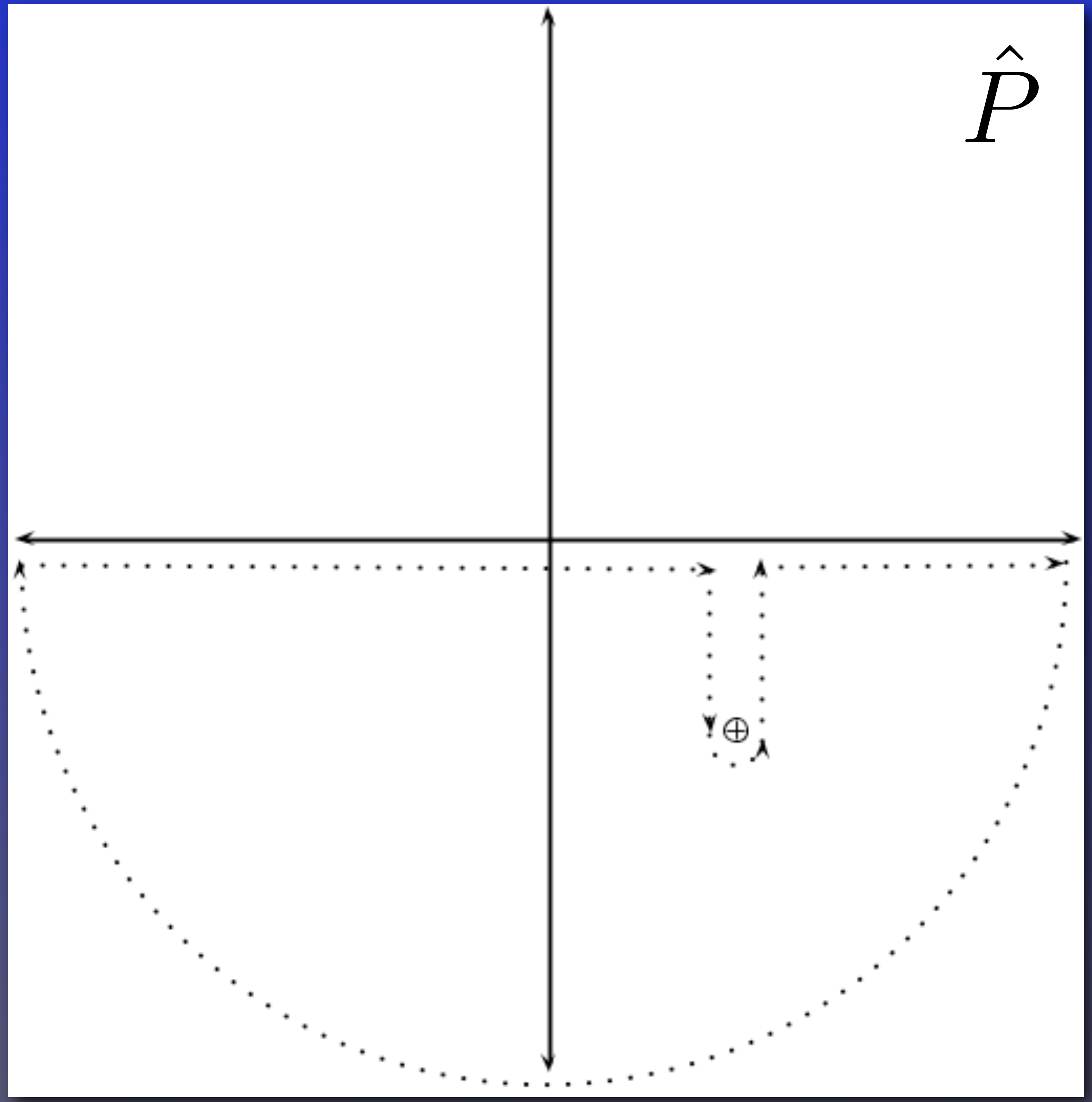
$$Z - P = \frac{1}{2\pi} \oint d(Arg(w(z)))$$

FEL Dispersion Relation

FEL Dispersion Poles



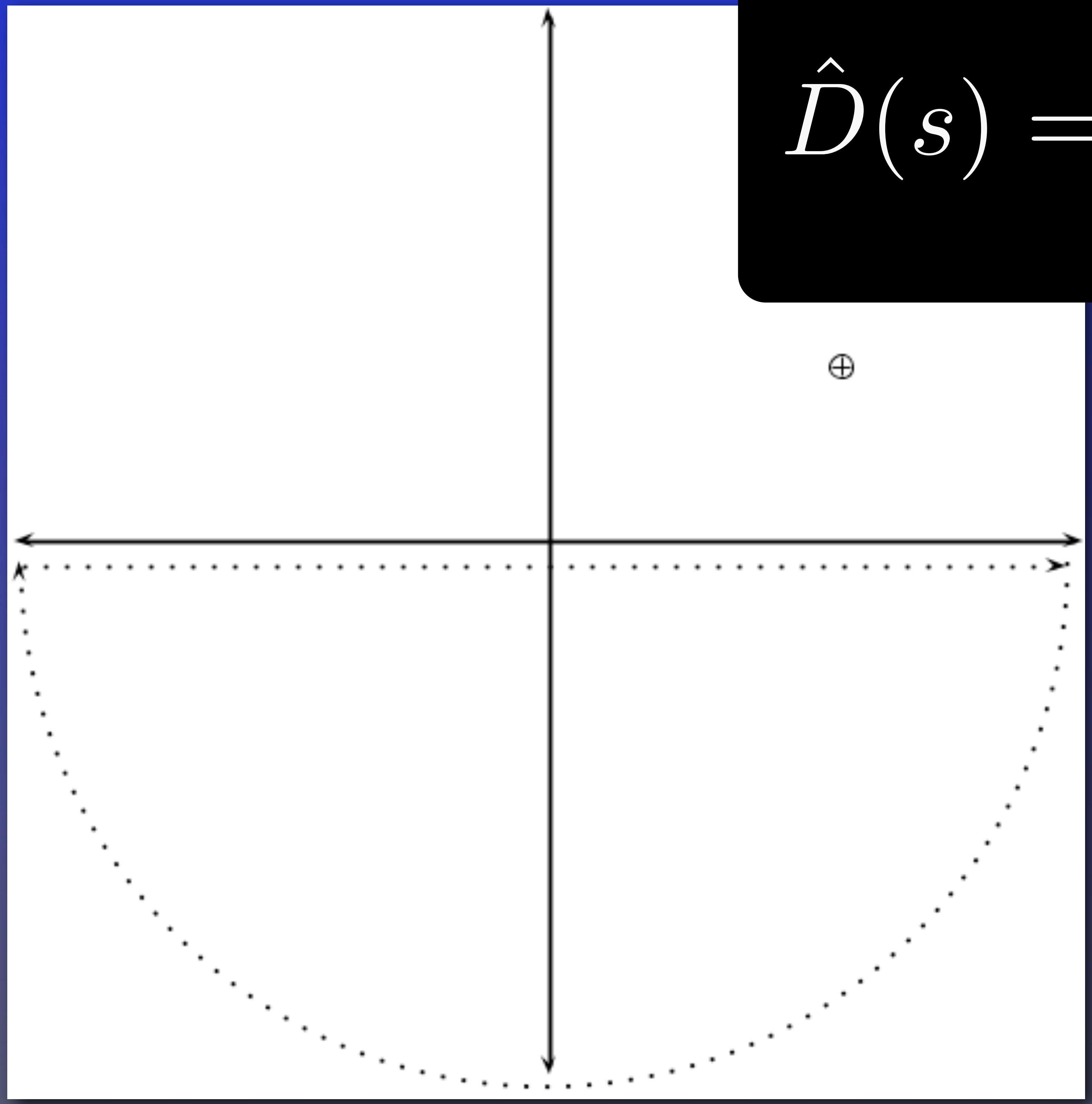
Landau
Prescription



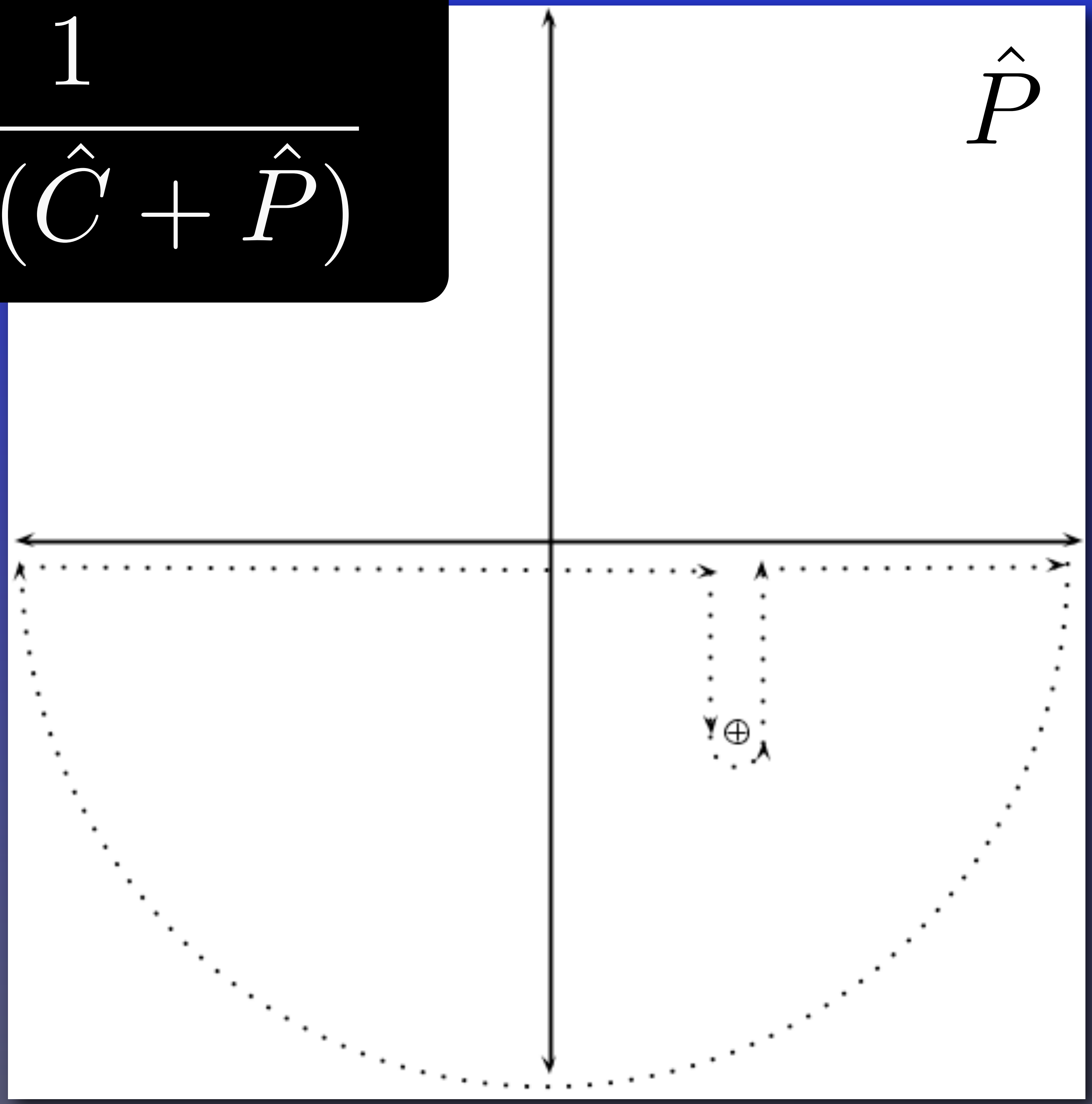
FEL Dispersion Relation

FEL Dispersion Poles

$$\hat{D}(s) = \int d\hat{P}\, \frac{d\hat{F}}{d\hat{P}} \frac{1}{s + \imath(\hat{C} + \hat{P})}$$

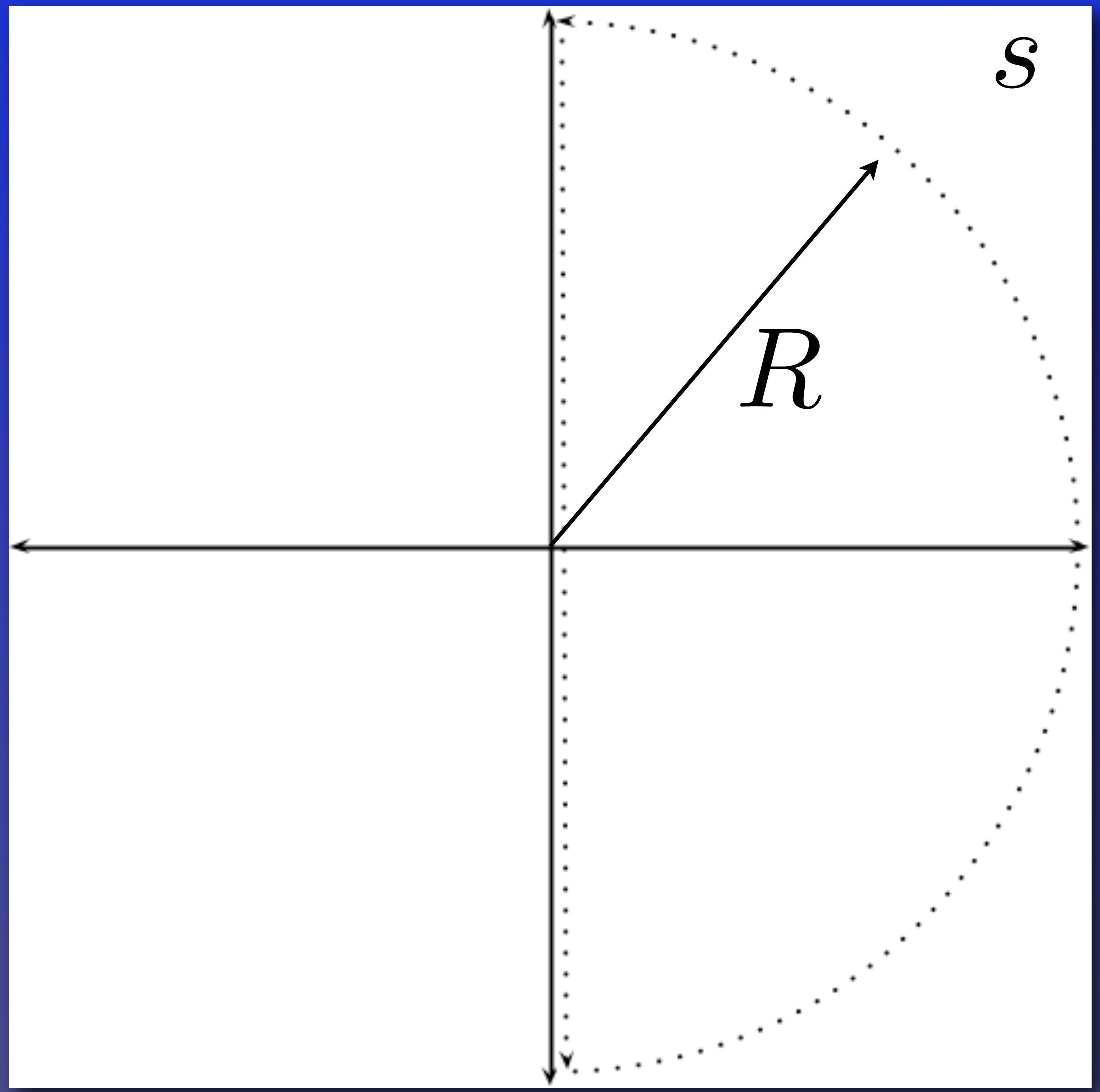


Landau
Prescription



FEL Dispersion Relation

FEL Amplifying Modes



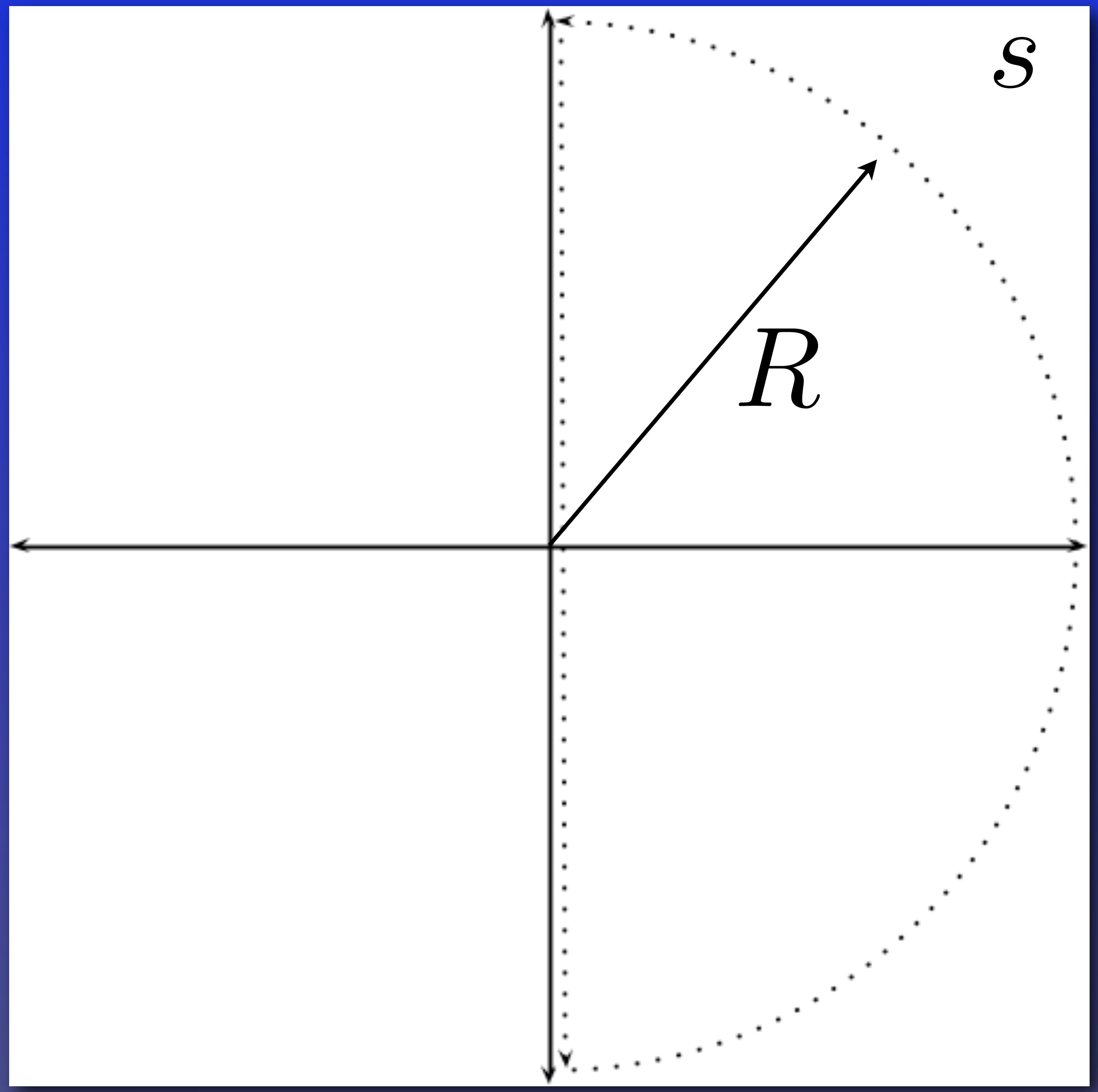
Parameterize Contour

$$s = Re^{\imath\theta} \quad \theta \in (-\pi/2, \pi/2)$$

$$s = \imath t \quad t \in (\infty, -\infty)$$

FEL Dispersion Relation

FEL Amplifying Modes



Parameterize Contour

$$s = Re^{\imath\theta} \quad \theta \in (-\pi/2, \pi/2)$$

$$w(s = Re^{\imath\theta}) = s + \mathcal{O}(R^{-2})$$

$$s = \imath t \quad t \in (\infty, -\infty)$$

$$w(s = \imath t) = \imath t - \int dP \; \imath \frac{d\hat{F}}{d\hat{P}} \frac{1}{t + \hat{C} + \hat{P}}$$

FEL Dispersion Relation

FEL Amplifying Modes

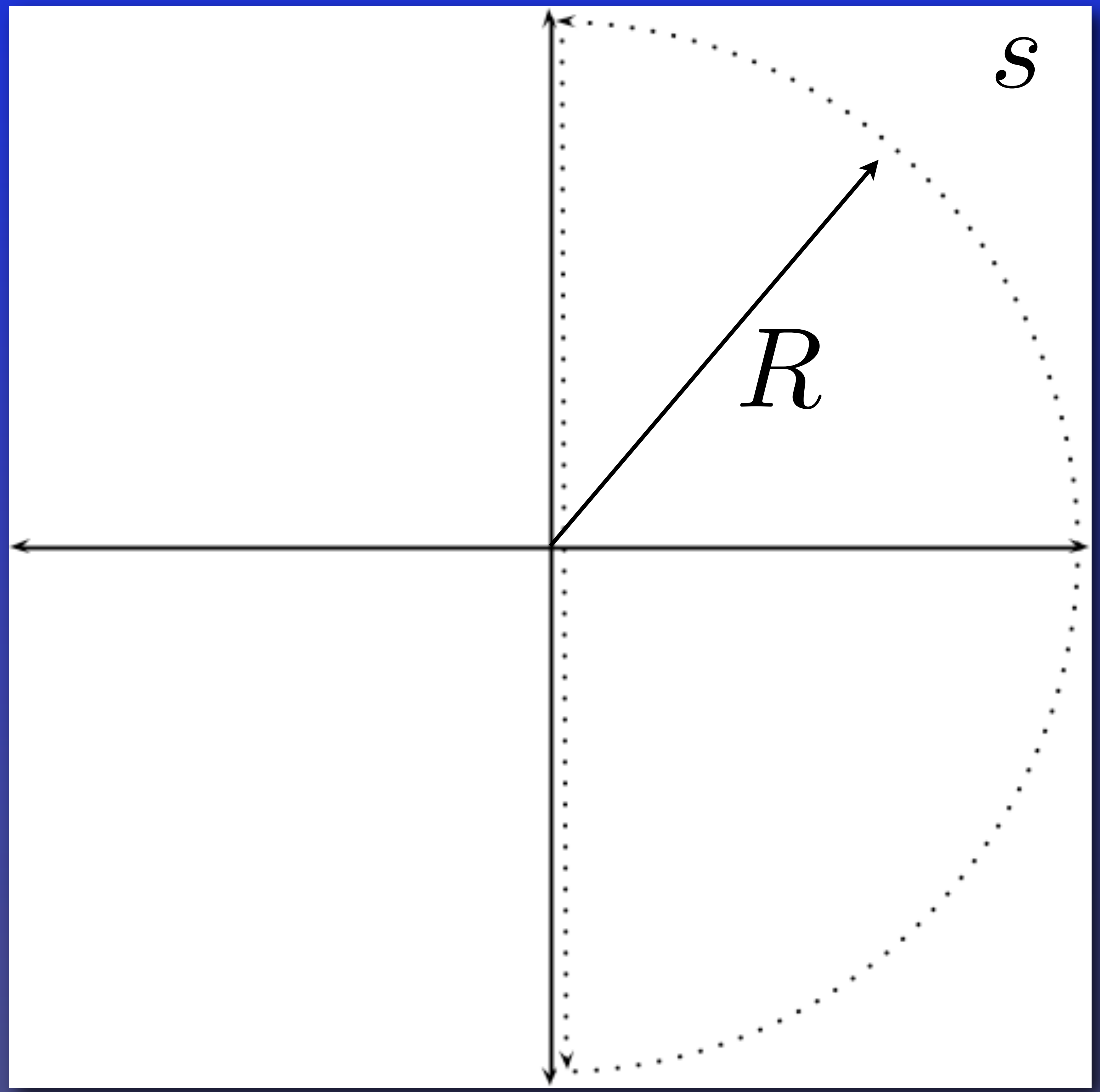
per Landau

$$\int dP \, \frac{d\hat{F}}{d\hat{P}} \frac{1}{t + \hat{C} + \hat{P}} = \mathcal{P} \int dP \, \frac{d\hat{F}}{d\hat{P}} \frac{1}{t + \hat{C} + \hat{P}} + \imath\pi \hat{F}'(\hat{P} = -t - \hat{C})$$

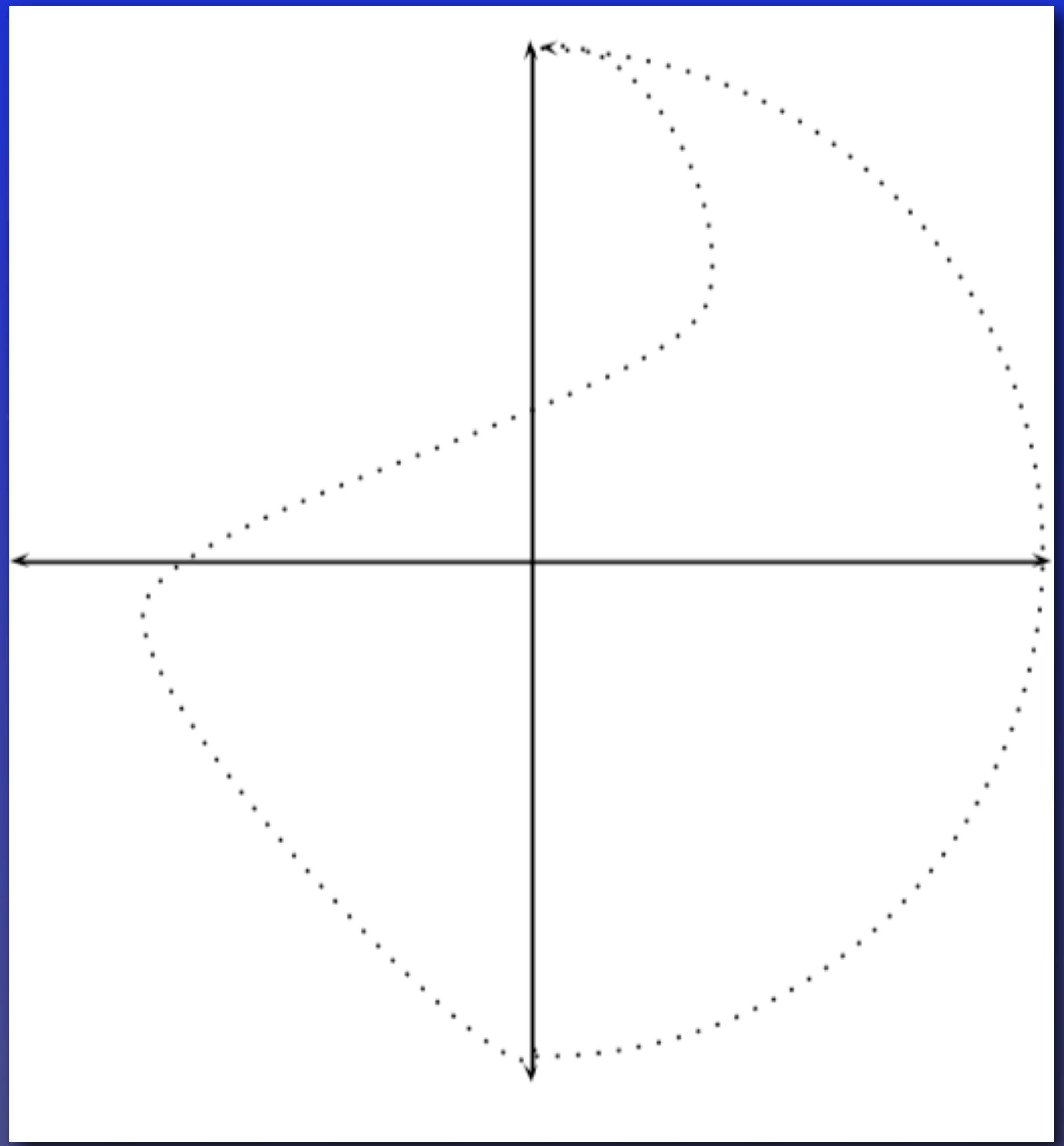
Bell curves only have one place where the imaginary part of this vanishes

FEL Dispersion Relation

FEL Amplifying Modes



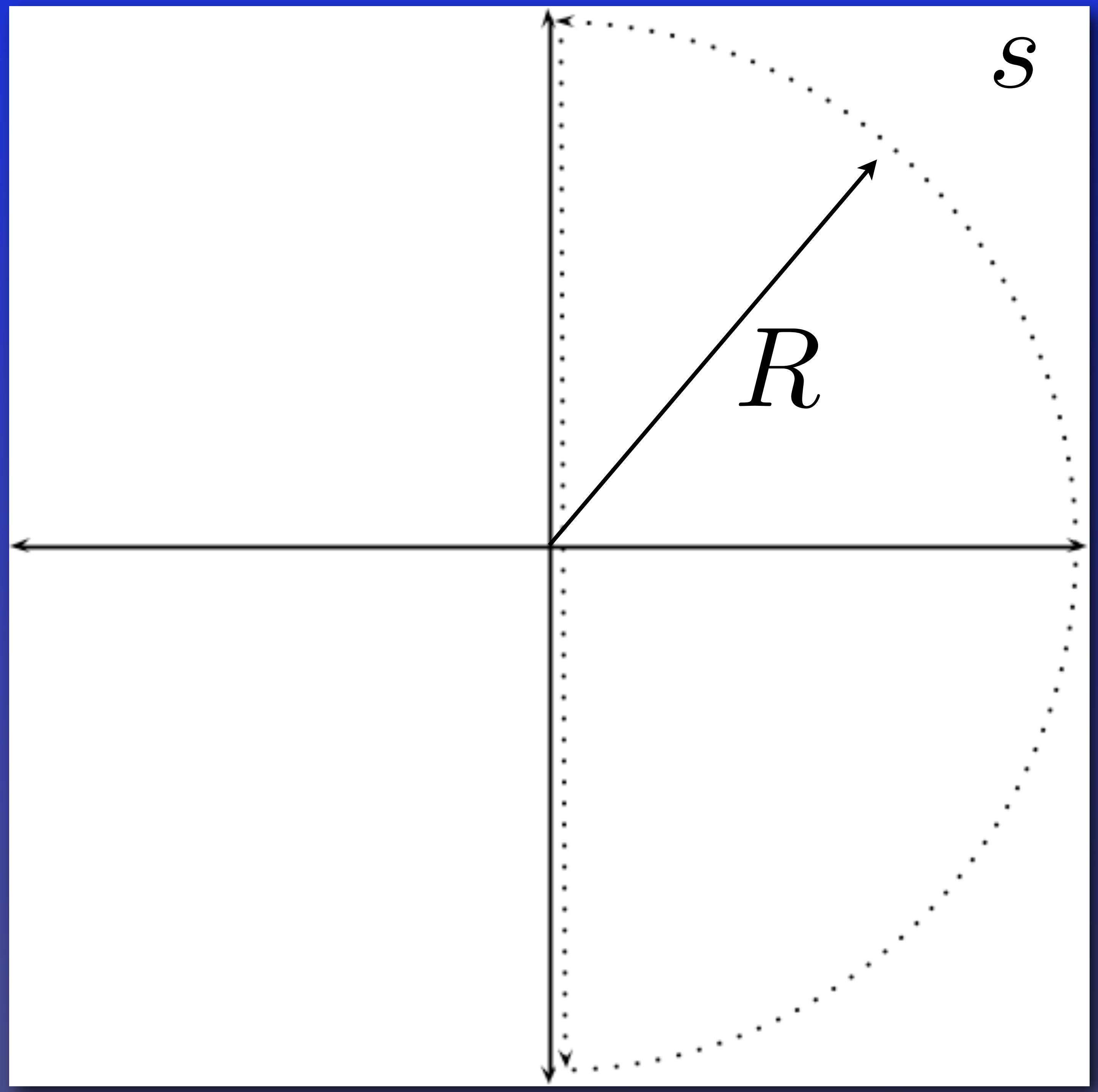
$$s - \hat{D}(s)$$



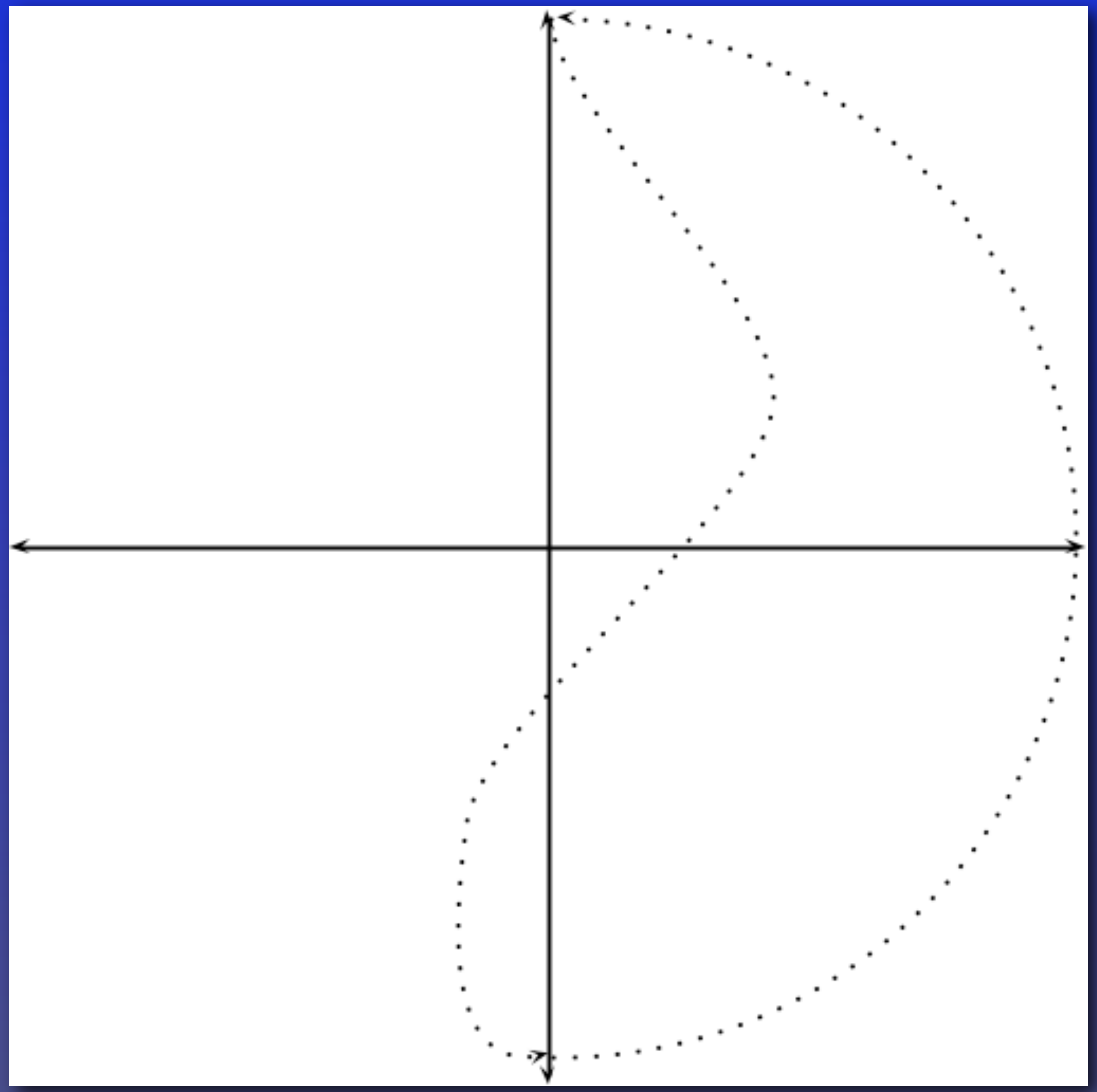
One growing mode

FEL Dispersion Relation

FEL Amplifying Modes



$s - \hat{D}(s)$



No growing modes

FEL Dispersion Relation

FEL Amplifying Modes

How does the image of the vertical line cross the imaginary axis?

Does this depend on the detuning?

FEL Dispersion Relation

FEL Amplifying Modes

Critical Frequency

For FEL dispersion relation

$$\hat{F}'(s = -t - \hat{C}) = 0$$

$$\hat{C}^* = \mathrm{Im}\left[\int d\hat{P}\, \frac{d\hat{F}}{d\hat{P}} \frac{1}{\imath \hat{P}}\right]$$

$$\mathrm{Im}\left(\imath t - \hat{D}(s = \imath t)\right) > 0$$ No growing mode

$$\mathrm{Im}\left(\imath t - \hat{D}(s = \imath t)\right) < 0$$ One growing mode

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